TRANSACTIONS OF
THE
OPTICAL SOCIETY

VOL. XXVIII. 1926-27 No. 5.

PRESIDENTIAL ADDRESS

SOME UNCULTIVATED OPTICAL FIELDS

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10th February, 1927.

THE PRESENT POSITION OF GEOMETRICAL OPTICS

WITHIN the whole domain of optics I suppose there is no region in which more of the present members of the Optical Society are vitally interested than that which we know as geometrical optics. It is a region in which many of these members have engaged with notable success. Resisting the natural, though as I think dangerous, precept to let well alone, I wish to consider with you the present status of geometrical optics with the express intention of seeing in what directions we may most profitably strive to advance.

A few years ago Professor Silvanus Thompson found it necessary to write of optical theory as it was then presented in this country:

> No doubt such books as Heath's *Geometrical Optics* and Parkinson's *Optics* are good in their way. They serve admirably to get up the subject for the Tripos; but they are far too academic, and too remote from the actual modern applications. In fact, the science of the best optical instrument-makers is far ahead of the science of the text-books. The article of Sir John Herschel "On Light" in the *Encyclopaedia Metropolitana* of 1840 marks the culminating point of English writers on optics.
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> The simple reason of the badness of almost all recent British text-books of optics is that, with the exception of one or two works on photographic optics, they are written from a totally false standpoint. They are written, not to teach the reader real optics, but to enable him to pass examinations set by non-optical examiners. The examination-curse lies over them all. Probably the reason why no English publisher has yet been found courageous enough to bring out translations of Czapski's *Optical Instruments* or Muller-Pouillet's *Optics* is that, even if translated, they would not command a large sale, because it would be useless for any student to cram himself up on them for an examination. The optical books which sell in England to-day are cram-books for university examinations. And so there is, to those who know, little inducement to write treatises upon real optics.

It will hardly be disputed that at the time the criticism was merited. To-day the position is entirely changed. We have available in English a number of books

on optics which set forth, apart from what is learnt from special calculations, "the science of the best instrument-makers." In spite of their many excellent qualities these books fail to satisfy the critical reader, and it is no surprise to find that their influence on university studies has been negligible. In fact they reveal that the science of the best instrument-maker is no science at all, but an art. In the exercise of this art Dennis Taylor has claimed that "during the last twenty or thirty years as great a number of new and successful optical combinations have been developed and brought out in this country as in all other countries put together, excepting in the case of microscope objectives," and to few who saw the great exhibition at the Optical Convention last year will the tribute come as a surprise. In justice to our achievements in the excepted class it ought perhaps to be added that a number of our ablest microscopists consider our best microscope objectives in advance of the best produced elsewhere. We may then conclude that in the practice of the art we have no present cause for dissatisfaction.

But we cannot rest satisfied without a science of optical instruments. The academic theory is in the position in which Silvanus Thompson found it and the alternative falls far short of what is required. That all is not as satisfactory as could be desired is well realised by our manufacturers, who, notwithstanding their apprehensions that success is not to be achieved, have welcomed the attempt to interest mathematicians in these optical problems. With what response that appeal has met I do not know; meanwhile we can at least consider whether we are not able ourselves to make some progress towards a happier state.

One of our main objects would be secured if we found a simple way of obtaining a general insight into the leading properties of lenses and series of lenses. The very title of our subject suggests that carefully drawn diagrams played an important part in the laying of its foundations, and a graphical method of exhibiting the main properties of lenses has much to commend it. The full possibilities of drawing as an aid in optical investigations have not been exhausted, and a method to be indicated presently may perhaps be found to give all that is required in the simpler problems, such as the determination of "best form" spectacles.

However successful drawing may be it will always require to be supplemented by other processes. There has been for some years past among our manufacturers a belief that the form of solution of lens problems most helpful to them would express the shape of a refracted wave in terms of the shape of the incident wave. In a disguised form—the disguise consisting in the simplification of removing the forms of the incident and emergent waves from the problem—this proposal was discussed at length about a century ago, a time when workers in geometrical optics in this country showed exceptional activity. The fact that on the foundation then laid no suitable edifice has been constructed has long been a standing reproach to this country. I have therefore taken this opportunity of discussing on this basis the elements of geometrical optics (though the term then becomes a misnomer from the point of view of the wave theory), in order to show that there is decided promise that an exact optical science can be erected on the foundations laid so long ago.

The long-continued neglect of Hamilton's work is the more extraordinary as some twenty years ago Lord Rayleigh specially commented on it in one of his addresses as President of the Royal Society. His criticism still appears so pertinent, and the remedy he suggests, though clearly practicable and sufficient, is so rarely administered, that his remarks will bear repetition at length as advice to those who would assist in the advancement of optics at the present day. Lord Rayleigh said:

> For the advancement of science the main requirement is, of course, original work of a high standard, adequately explained and published. But this is not enough. The advances so made must be secured, and this can hardly be, unless they are appreciated by the scientific public. In some branches of Pure Mathematics it is said that readers are scarcer than writers. At any rate the history of science shows that important original work is liable to be overlooked and is perhaps the more liable the higher the degree of originality. The names of T. Young, Carnot, Waterston, and B. Stewart will suggest themselves to the physicist; and in other branches, doubtless, similar lists might be made of workers whose labours remained neglected for a shorter or a longer time. In looking into the more recent progress of Geometrical Optics, I have been astonished to find how little correlation there has been between the more important writings. That Coddington should have remained unknown in Germany and von Seidel in England need not greatly surprise us; but in this subject it would appear that a man cannot succeed in making even his own countrymen attend to him. Coddington seems to have heard nothing of Cotes and Smith, and Hamilton nothing of Airy and Coddington.
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> It is true that no two writers on theoretical subjects could differ more in taste and style than do Hamilton and Coddington. The latter addressed himself to special problems, the solution of which seemed to have practical importance. Among his achievements was the rule relating to the curvature of images*, generally known as Petzval's, although Petzval's work was of much later date. Hamilton, on the other hand, allowed his love of generality and of analytical developments to run away with him. In his *Memoir on Systems of Rays*, with its elaborate and rambling supplements, there is little to interest the practical optician, though the mark of genius is throughout apparent. It was only in two or three pages of a later paper that he applied his powerful methods to the real problems of Optics. As Finsterwalder has remarked, his "six radical constants of aberrations," expressing the general properties of a symmetrical instrument, are at once an anticipation and a generalisation of von Seidel's theorems. But the published work is the barest possible summary. If Hamilton had been endowed with any instinct for Optics proper, he could have developed these results into a treatise of first-class importance. In more recent times Hamilton's footsteps have been followed by Maxwell as well as by Thiesen and Bruns, of whom the two latter do not seem to have realised that Hamilton (or even Maxwell) had concerned himself with the subject at all....
>
> Enough has probably been said to illustrate my contention that much loss has ensued from ignorance and neglect of work already done. But is there any remedy? I think there ought to be. In all the principal countries of the world we have now a body of men professionally connected with science in its various departments. No doubt the attention of many of these is so engrossed by teaching that it would be hard to expect much more from them, though we must remember that teaching itself takes on a new life when touched with the spirit of original enquiry. But in the older Universities, at any rate, the advancement of science is one of the first duties of Professors. Actual additions to knowledge occupy here the first place. But there must be many who, from advancing years or for other reasons, find themselves unable to do much more work of this kind. It is these I would exhort that they may fulfil their function in another way. If each man

* The credit should have been given to Airy.

would mark out for himself a field—it need not be more than a small one—and make it his business to be thoroughly conversant with all things new and old that fall within it, the danger of which I have spoken would be largely obviated. A short paper, a letter to a scientific newspaper, or even conversation with friends and pupils, would rescue from oblivion writings that had been temporarily overlooked, thereby advancing knowledge generally and sometimes saving from discouragement an unknown worker capable of further achievements. Another service such experts might render would be to furnish advice to younger men desirous of pursuing their special subject.

It is particularly interesting in view of these remarks to find that the work of the greatest writers in English on optics fits together to make a harmonious whole. A solution of optical problems by the method of Hamilton, and cast in the form originated by Airy, leaves little to be desired. My present purpose is to show that by the development of the two distinct lines of investigation I have mentioned, the graphical and algebraic, there is yet hope that this optical art may be elevated to the status of a science.

GEOMETRICAL OPTICS TREATED GRAPHICALLY

Collinear Optics

The generally accepted treatment of optical instruments is based upon the supposition that the object and image surfaces which are projected sharply upon one another are planes normal to the axis of symmetry of the instrument. Fig. 1 shows such a system in section. The principal focal surfaces are the planes through F and F'; the unit planes pass through the unit points U and U'; and u and u'

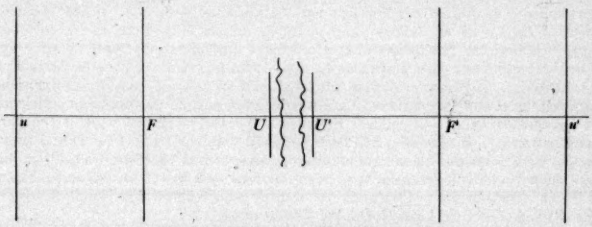

Fig. 1. Image surfaces of collinear optics

are the points in which the negative unit planes, where object and image are of the same size but inverted relatively to one another, meet the axis. This is essentially the picture of image formation inculcated by our text-books, and that which most users of optical instruments, such as photographers, assume to be tolerably correct. Fig. 2 shows a diagram of a type familiar to all of us in our student days. I have not repeated it here to revive unhappy memories, but to recall the assumption on which the system of images illustrated in Fig. 1 rests. This is a strict point to point correspondence between the object and image spaces; for instance, any ray such as GH, passing through the axial point at infinity in

the object space, will pass through F' on refraction and also through the point H' of the second unit surface obtained by annihilating the space between the two unit planes, so that H and H' become a single point; and the ray GU, through the combined unit and nodal point U, emerges through U' unaltered in direction.

Geometrically this system is elegant if somewhat trivial. Its chief merit—that

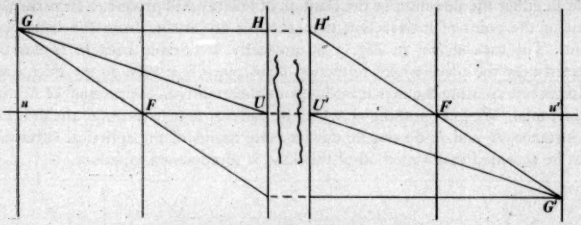

Fig. 2. Image construction of collinear optics

nothing need be known about the construction of the instrument—is discounted by our knowledge, as was indicated very simply by Maxwell, that the whole conception is inconsistent with the known properties of light. It is to be regretted that it occupies so much space in our text-books. If Maxwell's thorough but concise treatment were imitated, and the space thus saved devoted to the problems of "real" optics, there would be less justification for the prevailing view* that optics is a "dead" subject.

Real Optics

The single spherical surface

Although no purely geometrical treatment can be considered entirely adequate for the discussion of optical problems, there are such decided advantages in a method which presents an intelligible picture to the eye that the composition of a substitute for the incorrect "collinear" diagram is of no little importance. The most obvious course would be to trace a number of rays graphically through the system; even were there no other objections it is probable that diagrams constructed in this way would fail to achieve their object owing to the difficulty of following out correctly a particular group of rays which intersects other groups shown on the same diagram. The method however suffers from the fundamental defect that the paths cannot be traced graphically with the accuracy necessary to convey useful information about the image. The old collinear diagrams such as Fig. 2, it may be recalled, only achieve their aim because they are violently distorted, distances perpendicular to the axis being greatly exaggerated in comparison with those parallel thereto. It is one of the peculiarities of collinear imagery that this distortion introduces no error into the resulting picture, but this property no longer

* Professor Filon, *Trans. Opt. Soc.* **22** 1921 (208).

holds when we wish to investigate by geometrical drawing that type of imagery which follows from the sine law of refraction. In Fig. 3 are shown two rays *PH* and *pH* lying in the same plane which after refraction by a spherical surface *OpP* lie along *PK* and *pK*. Almost invariably in optical instruments the angles between the rays which form the image of any particular point are small, and a very small error in either the direction or the position of one ray will produce a large displacement of the point of intersection with another ray arising from the same object point. The case shown in Fig. 3 is unusually favourable because *H* has been selected near the place where refraction takes place, but even so the thickness of the lines representing the rays introduces an uncertainty in the position of *K* which is quite fatal. We have nothing on which we can rely to help us out of the difficulty: for instance, *H* and *K* do not lie on the same radius of the spherical surface. It must be accepted that a method of this kind is foredoomed to failure.

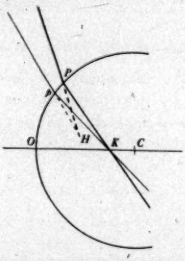

Fig. 3. Graphical construction in primary plane for a refracting sphere

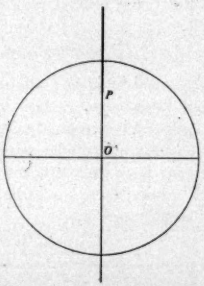

Fig. 4. End view of refracting sphere: ray plane vertical

It is a convenient assumption that the diagrams we construct showing the paths of rays represent a vertical section through the instrument, the axis of symmetry usually appearing as a horizontal line. Such a diagram as Fig. 3 then represents the paths of rays lying in a vertical plane, the axis for the moment, since the surface is spherical, being the horizontal radius in the plane. Thus the end view of the system shown in Fig. 3 will appear as Fig. 4, *P* being the point where refraction takes place and *O* the point of intersection of the axis with the surface. We shall obtain a fresh (vertical) plane *BOB'* for our diagram if we suppose the system of Figs. 3 and 4 rotated about the axis *OC* into such a position as is shown by the end view, Fig. 5. In this new plane the rays *PH*, *PK* of Fig. 3 are represented by the two points *I* and *J* (Fig. 6) in which they meet the axis *OC*. Now *I* and *J* are conjugate points, though in a different sense from the pair *H* and *K* of Fig. 3. In the accepted nomenclature of our subject, *H* and *K* are primary conjugate foci, and *I* and *J* are secondary conjugate foci. The latter are defined as the points in which the incident and emergent portions of any ray meet a plane

containing the axis of the optical system. For a single spherical surface this plane may be replaced by any plane passing through the centre of the sphere.

We obtain a clearer picture by rotating the sphere together with the plane *POC* about the vertical axis *BB'* (Fig. 5) until the plane *BAB'* passes through the observer's eye. The new view is shown in Fig. 7. The incident and refracted rays are shown

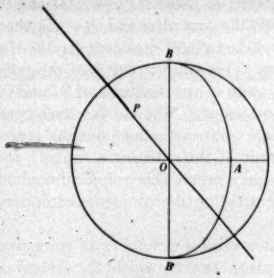

Fig. 5. End view of refracting sphere: ray plane inclined to vertical

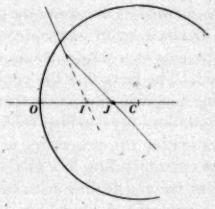

Fig. 6. Secondary conjugate points

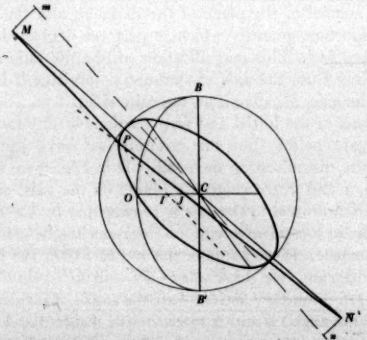

Fig. 7. Oblique view of secondary conjugate points

meeting *OC* in *I* and *J*, the former being a virtual focus. If we rotate the plane *POC* about *OC*, it is evident that the points *I* and *J* are conjugate foci for an infinite number of rays refracted similarly to the one shown. The condensation of rays at these secondary foci is obviously much greater than at the primary foci *H* and *K* of Fig. 3, and the secondary points are of correspondingly greater importance. In a similar way if any other plane through *C* is selected its points

of intersection with the rays, say M and N, will be a pair of secondary conjugate foci, and will be common to all the rays obtained by rotating the plane containing the rays about MN.

We may now rotate the plane containing MPN about any axis through C, and obtain a new position mCn for MCN. From symmetry m and n will then be conjugate secondary foci for a set of rays corresponding to those for which M and N are conjugate. For suitably defined sets of rays the arcs Mm and Nn will then bear an object-image relation to one another, and the ratio of their lengths Nn/Mm may be defined as the secondary magnification. This ratio is equal to the ratio of the distances of M and N from C, or to the ratio of the distances of M and N from the axis by rotation about which the displacements Mm and Nn have been obtained. The facts that the conjugate points are collinear with C and the corresponding magnification is determined by the ratio of their distances from C, are the foundations upon which we are about to build a practicable graphical method of investigating the properties of lenses by constructing the conjugate secondary surfaces corresponding to a given magnification.

When we are dealing with an instrument comprising a number of refracting surfaces any axis about which rotation takes place, and to which the foregoing considerations are to be applied, must be common to all the surfaces. That is to say the properties found for an individual surface will apply to a complete instrument if the rotation takes place about the axis of symmetry. The displacements Mm and Nn are then normal to the plane of the diagram, and the magnification will denote a most important quantity which is perforce neglected in diagrams constructed for primary foci. This magnification will be the ratio of the distances of corresponding points from the axis of symmetry—positive if both points are on the same side of the axis, negative if on opposite sides.

The intersections of the initial and final portions of any ray with the axis are themselves conjugate points, but the criterion we have just reached for the determination of the magnification then fails. If in Fig. 7 we suppose OC is the axis of the system, I and J are conjugate points on the axis, and we have found that the magnification normal to the axis is represented by the ratio $CJ : CI$. We wish to express this in a form involving no reference to the centre of curvature C of an individual surface. If we suppose the radius PC at the point of refraction inserted in the diagram, we have $CI : CP = \sin CPI : \sin CIP$ and $CJ : CP = \sin CPJ : \sin CJP$. Now CPI and CPJ are the angles of incidence and refraction, and CIP and CJP the angles ψ and ψ' made by the incident and refracted portions of the ray with the axis of symmetry. It follows, if μ and μ' are the refractive indices, that the secondary magnification is equal to $\mu \sin \psi : \mu' \sin \psi'$, and very elementary considerations enable us to see that a relation of this form holds for a complete instrument which has the centres of curvature of all its surfaces lying on a common axis. We therefore have convenient means of determining the magnification at all parts of the field. Conversely, if the secondary image surfaces for an assigned magnification are given, we can mark off the pairs of points which are conjugate to one another.

We may now pass on to consider the form of the constant magnification surfaces formed by refraction at a given point of a refracting sphere. Referring to Fig. 8, let the incident and refracted rays MPN be met by a plane through C, the centre of the refracting sphere, at M and N. Through M and N draw MA, ND parallel to PN, PM to meet PC in A and D, and also MB and NE meeting PC in B and E, making the angles PMB and DNE equal to CPN, and consequently the angles AMB, PNE supplementary to MPC. Evidently MB and NE are parallel to one another, and the ratio $NE : BM$ is equal to $NC : CM$, that is equal to the secondary magnification. We now proceed to show that the points B and E and the lengths BM, NE are determined by the magnification. From the similarity of the figures $ACPBM$ and $PCDEN$ it is only necessary to prove this for one of them, say the

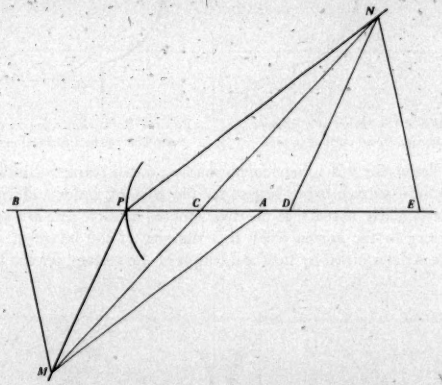

Fig. 8. Construction for secondary image surfaces

point E and the length NE. Since by the similar triangles PMC, CND the magnification is equal to the ratio $CD : PC$, D is a fixed point on PC. Also from the similar triangles PNE, NDE

$$\frac{PE}{NE} = \frac{NE}{DE} = \frac{\sin PNE}{\sin NPE} = \frac{\mu'}{\mu},$$

or $PE/DE = \mu'^2/\mu^2$, so that E is a fixed point on PC; hence also NE is a fixed length. Thus the locus of N is a sphere having its centre at E, and similarly M lies on a sphere having its centre at B. The complete secondary surfaces corresponding to any assigned magnification can therefore be drawn easily and with great accuracy.

Among the most important of the constant magnification surfaces are the principal focal surfaces corresponding to the magnifications ∞ and 0. The conjugate surfaces are at infinity, and the points A and D in the two respective cases coincide with C. Fig. 9 shows a section of a refracting sphere with its centre at C. On the left of the thick line marking its boundary the medium is assumed to be air,

and, on the right, glass of refractive index 1·5. For light refracted at V the principal focal surface in air is the sphere shown meeting CV at F. The effective portion of the surface is obviously limited by the tangent plane to the refracting surface at V. Fig. 10 shows the principal focal surface in glass for the same sphere. This surface is limited by a plane though C normal to the axis VC. In Fig. 11 four pairs of focal

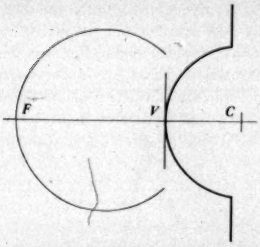

Fig. 9. Refraction at the vertex of a sphere: secondary principal focal surface in air

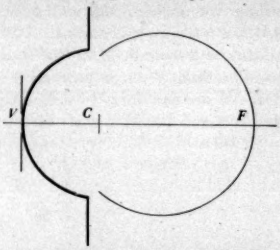

Fig. 10. Refraction at the vertex of a sphere: secondary principal focal surface in glass

surfaces are drawn for this sphere corresponding to the refractive indices 1·4, 1·5, 1·6, 1·7, the focal surface being largest for the smallest index and vice versa. It should be particularly noted that the two spheres of each pair are of equal size. This is contrary to the normal view that the size of the image of an infinitely distant object is determined by the focal length of the system, for the focal lengths

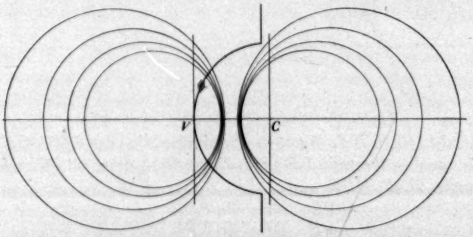

Fig. 11. Refraction at the vertex of a sphere: secondary principal focal surfaces for refractive indices 1·4, 1·5, 1·6, 1·7

of the surface in glass and in air are in the ratio $\mu : 1$. Against this we have to place the fact that the solid angles which the infinitely distant projections of these surfaces subtend at V are different. Another point of interest is that the focal surfaces for all refractive indices form a coaxal system of spheres with V and C as limiting points. Thus the system of air surfaces is situated with respect to V in precisely the same way as that in glass is situated with respect to C. This is a matter to which we shall return later.

In Fig. 12 the same refracting sphere is shown but the focal surfaces FG are only drawn for a single refractive index. The unit surfaces of course both coincide with the refracting surface itself. If we take centres on VC twice as far from V as the centres of the focal surfaces FG, and describe spheres about these new centres with double the radius of the focal surfaces, we obtain the negative unit surfaces UW.

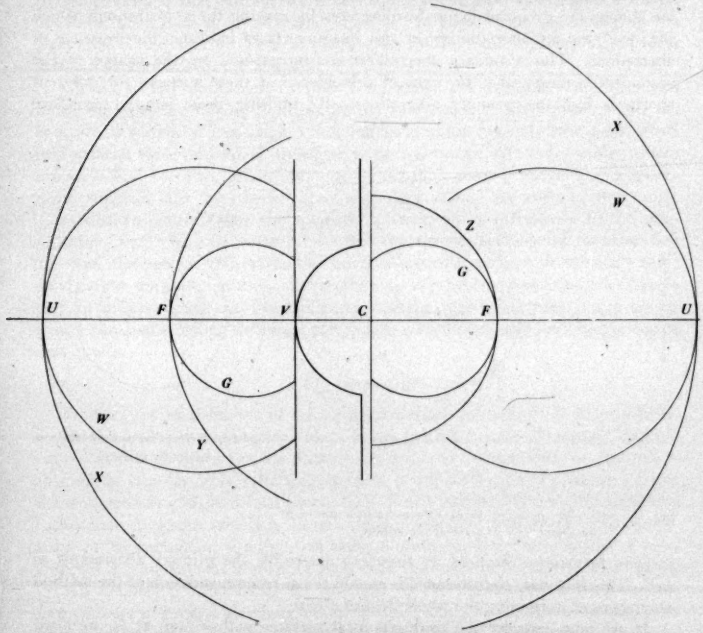

Fig. 12. Refraction at the vertex and at the centre of a sphere: secondary focal and negative unit surfaces

These of course are of equal size by definition. The principle governing the spacing of the whole series of spheres is now obvious. The same proportion of each sphere is effective, the limiting circles lying on a plane through V in the case of the air surfaces and on a cone having its vertex at V and the critical angle for its semivertical angle in the case of the glass surfaces. A certain dissimilarity is thus introduced into the two series of surfaces for varying magnification owing to the difference between the two focal lengths of the refracting surface. The plane

of symmetry for the principal focal surfaces bisects VC; that for the unit surfaces passes through C.

Let us now suppose that, instead of limiting the rays by assuming them to be refracted in the neighbourhood of V, we assume that they are limited to pass close to C. From the symmetry of the refracting sphere with respect to C we see that the unit surfaces are now spheres centered at C with radius CU. The fact that as we change the group of image-forming rays by moving the stop through which the rays pass we alter the object and image surfaces indicates the presence of aberrations. The secondary aberrations are represented by the shapes of the secondary surfaces, and the want of coincidence of these surfaces for different apertures indicates primary aberrations. By utilising these simple secondary

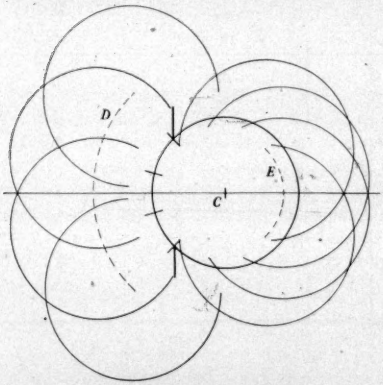

Fig. 13. Secondary principal focal surfaces for four apertures on the surface of a refracting sphere

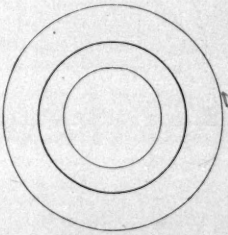

Fig. 14. Refracting sphere and its aplanatic surfaces

surfaces to explore the field we therefore determine the primary aberrations as well. This fortunate circumstance is an additional recommendation of the method as a means of examining the whole field of a lens.

If we now consider the principal focal surfaces with a stop at C, we shall again obtain spheres passing through F, F with C as centre, but as the two foci are at different distances from the centre the new focal surfaces are no longer equal. The focal surface in air FY has its radius μ times as great as the focal surface FZ in glass. This change in dimensions, it is to be observed, is in the opposite direction to that which might be anticipated as a result of the difference in focal length. The larger surface is formed in the medium with the smaller focal length and is the conjugate of a smaller portion of the infinite sphere than the smaller focal surface. The extent of the primary aberrations is measured directly by the separation of the surfaces FG and FY or FG and FZ as the case may be.

The primary aberrations are shown in a rather different form in Fig. 13. The refracting surface consists of a portion of the sphere, the aperture subtending a right angle at C. The principal focal spheres are drawn for refraction at four equally spaced points on a principal section, the azimuth at C changing by 30° between successive points. Since the centres of the air surfaces lie on a surface D farther from C than the surface E containing the centres of the glass surfaces, the linear aberration in air is greater than that in glass. This superiority on the glass side of the aperture is clearly conveyed by the closer spacing of the focal surfaces on this side in the diagram.

Clearly in any optical system free from primary aberrations the object and image surfaces for the same magnification must occupy fixed positions as the aperture is varied. For a single refracting sphere this can only take place when the surfaces, since they are all spherical, have C as centre. We then get the system shown in Fig. 14. The surface indicated by the thick circle is the refracting sphere itself, which is also the self-conjugate secondary surface for magnification unity. The other two surfaces are the aplanatic spheres, which correspond to the magnification $1/\mu^2$. We may further note that the various surfaces for magnification $1/\mu$ pass through C. Wherever the surfaces coincide, or pass through the same axial points, the sine condition will be satisfied. We therefore see from this example that an optical system may have as many as three pairs of aplanatic points on the axis, a conclusion which definitely contradicts a statement made in most advanced text-books.

The simple lens

In this brief examination of refraction at a single surface we have been led to regard the sphere rather than the plane as the normal surface of constant magnification. Although this is not the occasion on which to examine fully to what extent this will be true for other refracting systems, it will not be out of place to consider the corresponding surfaces for a simple lens. This is particularly desirable as the lens rather than the single surface is the natural unit in optical instruments, and a knowledge of the behaviour of a lens will assist in understanding the behaviour of more complex instruments far more effectively than the study of a single surface can do.

All ordinary positive lenses are bounded by two intersecting spheres, but the two surfaces in general give four lenses of which only one is likely to be employed in real instruments. In Fig. 15 the four lenses formed by spheres having their two radii and the distance between their centres of curvature in the ratios 5 : 10 : 6 are shown. The first of these, a, characterised by its slenderness, is of somewhat extreme shape judged by those commonly employed in instruments, and the three remaining forms b, c, d, would probably never be given to real lenses at all. They are sufficiently abnormal for us to have some difficulty in judging of their relative powers. With a refractive index of 1·5 these are, taking the forms a, b, c, d in order, in the ratios $4 : 6 : 9 : -1$. Despite their great differences they have a number of common characteristics. What these are can be inferred from the following investigation in which the form a is illustrated.

Fig. 16 shows in section the two spheres intersecting at A and B, their centres being c_1 and c_2. The points R and S divide c_1c_2 internally and externally in the ratio of the two radii, and on RS as diameter a sphere is described. This sphere has several extremely important properties, some of which we will discuss briefly.

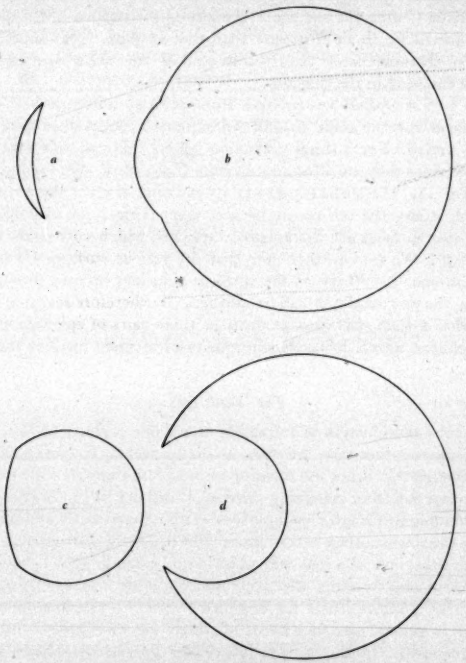

Fig. 15. Lenses constructed from two spheres

We will name it the harmonic sphere since the four points R, c_1, S, c_2 form a harmonic range. The three radii are not however in harmonic progression—the radius of the harmonic sphere is only half the harmonic mean between the two radii r_1 and r_2 of the lens surfaces when the axial thickness is zero.

In the first place we will show that, as drawn in Fig. 16, the harmonic sphere passes through the circular rim AB of the four lenses of Fig. 15. Referring to Fig. 17, since by construction $c_1S : Sc_2 = c_1A : c_2A$, AS is the internal bisector of

the angle c_1Ac_2; and since $Rc_1 : Rc_2 = c_1A : c_2A$, AR is the external bisector of the same angle. Hence RAS is a right angle and A lies on the sphere.

If any straight line through R meets the lens surfaces in I and J, the triangles RIc_1 and RJc_2 are similar. Thus any ray travelling within the lens in a direction

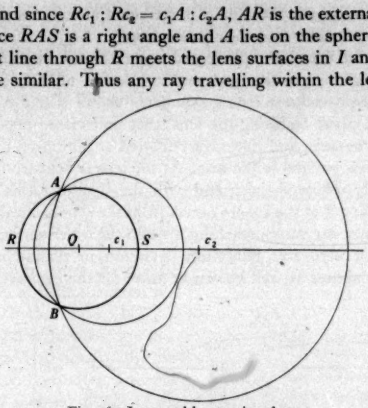

Fig. 16. Lens and harmonic sphere

passing through R meets both surfaces at equal angles, and the corresponding external rays are parallel. The images of R in the two surfaces are therefore nodal points. It must be observed that these are not assumed to be fixed points—they will in general depend on the direction of the ray.

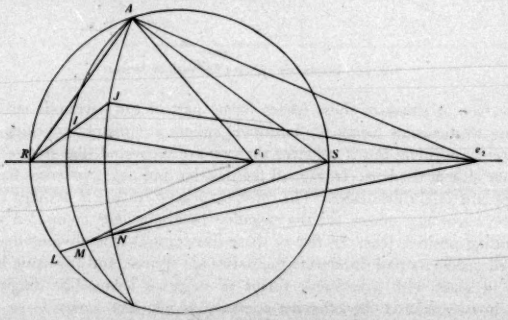

Fig. 17. Properties of harmonic sphere

Let us now consider a ray within the glass which passes through S. Let it meet the harmonic sphere in L and the refracting surfaces in M and N. Since $Mc_1 : c_1S = Nc_2 : Sc_2$, and the two angles at S are supplementary, the angles c_1MS and SNc_2 are equal. That is to say the two angles of refraction on the glass side are equal but lie on opposite sides of the normals. In other words a ray which

follows in the glass a path through S is refracted at minimum deviation. Such rays are not likely to suffer from heavy aberration.

Fig. 18 shows the harmonic sphere and rays refracted without deviation and at minimum deviation for positive and negative lenses of different shapes. With equi-convex and equi-concave lenses symmetry shows that the harmonic sphere degenerates into a plane bisecting the lens, that undeviated rays pass through its intersection with the axis, and that rays refracted at minimum deviation travel in the body of the glass parallel to the axis. As the lens is bent from the symmetrical form the harmonic sphere shrinks, and with the slender forms universal in real instruments the point R of the sphere moves from the symmetrical position towards the surface of greater curvature, reaching it when the other surface becomes plane. The conditions for zero and minimum deviation in plano-convex and plano-concave lenses are almost as well known as those for the symmetrical forms. The

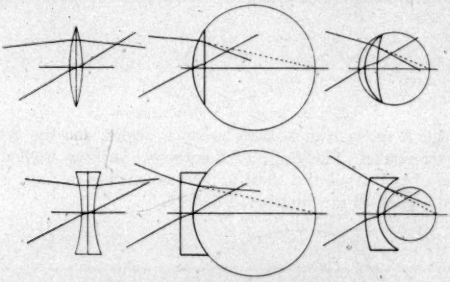

Fig. 18. Harmonic spheres for various lenses

curved surface of either of these lenses forms part of the harmonic sphere. On proceeding to meniscus forms the harmonic sphere completely envelops the lens if the latter is positive, but if negative the two are separated, the sphere lying on the concave side of the lens. In Fig. 18 the positive and negative lenses have equal curvatures and axial thicknesses. The most surprising feature is perhaps the small size of the harmonic sphere for the negative meniscus lens compared with the corresponding positive lens. In five of these illustrations it is obvious that no ray is refracted unless its path intersects the harmonic sphere, and the same holds for the path in glass with practicable forms of negative lens. The diagram thus indicates how restricted the effective aperture of negative lenses is—a fact not ordinarily appreciated because the surfaces do not intersect.

Reverting to Fig. 17, since the distances of any point on the sphere from c_1 and c_2 are in the ratio of the radii, it follows from a result we have already obtained that the triangles LMc_1 and LNc_2 are similar, so that the distances of L from the two surfaces measured along any ray passing through S are also in the ratio of the radii. In particular this is true for the point R. Under the name of the "centre"

of the lens—not a very appropriate term—this property is often used to define it. Obviously either R or S is the "centre" of each of the forms a, b, c, d. On paying attention to the signs of the radii in these various forms it is seen that S is the "centre" for forms b and c, and R for forms a and d. The latter result appears surprising enough, but not less so is the inference we have to draw that while lens a is to be regarded as "bent" to the left, lens b must be considered to be "bent" to the right. That is to say a positive lens is not necessarily bent towards the surface of greater curvature.

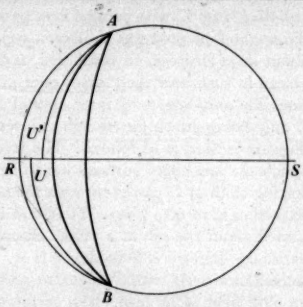

Fig. 19. Secondary unit surfaces and harmonic sphere

Resuming however, since the ray LS is refracted equally at both surfaces, and since the ray bisects the angle c_1Lc_2, the triangles LMm, LNn formed by Lc_1, Lc_2, LS, and the two portions of the ray in the air, are similar. Moreover, since the ratio of LM to LN is equal to that of the radii, we have $Lm : Ln = Lc_1 : Lc_2$, or mn is parallel to the axis. From the construction m and n are the images of L in the two surfaces. It follows that these images of the portion ARB of the harmonic sphere are the unit surfaces. They are shown in Fig. 19. Though drawn on a large scale the departures from the spherical form were found inappreciable.

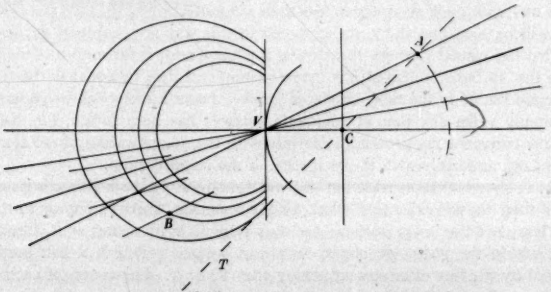

Fig. 20. Chart for constructing conjugate secondary surfaces

On account of these and other properties of the harmonic sphere, it is convenient to define by a stop situated on this sphere the light for which surfaces of constant magnification are to be constructed. The various rays will then be refracted at different points of the two lens surfaces, and a simple method of construction is required. A method which has been found to work well depends on

the construction on tracing cloth of a diagram represented in principle in Fig. 20. This consists essentially of one set of the circles shown in Fig. 11, drawn for a refracting surface of the correct radius, together with a number of straight lines radiating from V, with parallel lines passing through C. To secure close parallelism the method adopted was to draw a series of radiating lines at successive angles of about three degrees, on cardboard, and to trace these on the cloth with V and C placed in turn over the radiant point, the axis in both positions resting on a base line. For simplicity only three lines of each set are drawn in Fig. 20, those through C only being drawn far enough to intersect the circles. The method in which the diagram is used is as follows. The tracing is laid on the drawing of the lens for which the secondary surfaces are to be constructed. A drawing pin piercing the tracing cloth at C passes through the centre of curvature of the surface at which refraction is to take place. To obtain different points on the surface the cloth is turned about the pin as a pivot. Suppose the position of the stop for which the particular diagram is to be drawn is A. As the cloth is moved about C the point A traces out a circle with C as centre, and each of the radial lines through V may be brought over A in turn. Fig. 20 shows rotation from the symmetrical position through the angle VCT, bringing A over the third of the oblique radiating lines. If the refractive index of the glass is 1·5, the value for which the second largest circle has been drawn, a point on the surface required lies under B, the point in which the circle is intersected by the third line radiating from C. A record of the position of this point is obtained by pricking through the tracing on to the drawing below. The cloth is then rotated farther until the next line passes over A, when another point is pricked through, and so on. A complete surface may be traced in this way with good accuracy in less than a minute.

It will be seen that the surfaces traced in this way are conjugate surfaces for light that has passed through the glass as a plane beam; if the angles of incidence on the two surfaces are equal, the magnification will thus be equal to the ratio of the second radius to the first. Surfaces for any magnifications can be constructed very simply when any pair of conjugate surfaces has been found, for distances along the emergent ray proceed uniformly with the magnification, those along the incident ray uniformly with the reciprocal of the magnification.

Fig. 21 shows surfaces traced in this way for lens *a*. As both refracting surfaces present their convex sides to the left, all the secondary surfaces appear to the left of the lens, and the image surfaces are thus virtual. With a stop at R, situated of course within the glass, we obtain an object surface of which a half section is indicated by the line of circles stretching from D to E. The centre of each circle is a point pricked through on to the diagram in the way just explained. The corresponding virtual image surface is represented by GH. The two are symmetrically situated about R and the magnification is two, this being the ratio of the radius of the second surface to that of the first. These surfaces depart very slightly from the spherical form until exceptionally large angles of incidence are reached. To show this more clearly a circle has been added for the line GH.

If we now take S as the position of the stop within the glass we obtain the

object surface represented by the line of circles *DN* and the image surface *GMT*. These surfaces again are symmetrically situated with respect to *R*, and corresponding points are collinear with *R*; for instance, *M* is the image of *N*. The magnification for these surfaces is of course two, just as with the stop at *R*. The circles shown, having their centres on the axis, again indicate how slight are the departures of the secondary image surfaces from the spherical form. The departures found are due to the finite thickness of the lens, and diminish as the aperture is moved towards the margins of the lens where the thickness is less. When the aperture is

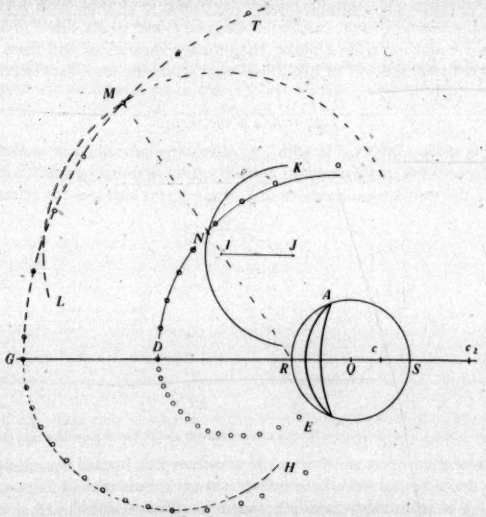

Fig. 21. Meniscus lens: various conjugate secondary surfaces

placed where the two surfaces meet the harmonic sphere, say at *A*, the conjugate surfaces found in this way become strictly spherical. The object surface is shown at *NK* and the virtual image surface at *ML*. These are only fractions of the more extensive portions of the spheres found as the secondary surfaces for a single refracting surface. The reduction is due to some of the light refracted at one surface meeting the other within the total reflection zone; the parts of the surfaces not effective through this cause have been omitted. The curiously unsymmetrical positions assumed by the segments will be noticed. Their centres *I* and *J*, it may be shown, lie at the same distance from the axis. This in itself, quite apart from the want of symmetry, shows that the magnification varies as we proceed along the surface. The derivation of the conjugate surfaces for constant magnification is

simple if it is observed that corresponding points on these surfaces lie on parallel lines through c_1 and c_2 respectively.

There are many other points of interest in diagrams of this kind which can hardly be considered here. The important fact to notice is that, as with the single refracting surface, so also with the simple lens, we are much nearer the truth if we regard the normal shape of constant magnification surfaces to be spherical than if we assume them to be plane. If this conclusion is valid for optical instruments in general—and we may note that it has been shown to be true for telescope objectives*—appreciable simplifications may result in dealing with higher order aberrations, which will, in the case of the secondary aberrations, relate to departures from a sphere and not from a plane; the primary aberrations will most naturally be considered from the way in which they depend on the secondary aberrations.

Lens with a plane surface

There is at least one case in which an alternative procedure is needed. If one of the surfaces of the lens is plane the corresponding secondary surface is at infinity, and one of the well-known methods of plotting image surfaces for refraction at a

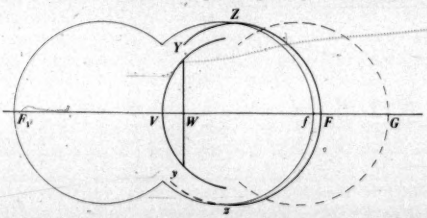

Fig. 22. Plano-convex lens: secondary focal surfaces for thin and thick lenses and single surface alone

plane must be used. Fig. 22 shows a plano-convex lens formed by adding a plane surface to the spherical refracting surface already considered in Figs. 9 and 10. The addition of this surface evidently makes no difference to the focal surface in air corresponding to refraction at the vertex V of the curved surface, but the other focal surface, which in the glass was situated at G (shown by a dotted line in Fig. 22), is now moved closer to the lens. If the thickness of the lens is negligible, this surface is simply translated without any change in its form or size through a distance equal to the radius of the refracting surface. The new surface is then indicated by the thin circle crossing the axis at f. The two focal surfaces are now symmetrical with respect to the tangent plane at V. When the lens is of finite thickness the focal surface on the plane side of the lens lies somewhat farther from V. For the lens terminated by the plane through W the focal surface has been carefully plotted and shown by the thick curved line FZY in the upper part of the diagram. Up to Z, the point of intersection of the f and F surfaces, the shape

* *Trans. Opt. Soc.* **25** 1923–4 (89).

hardly differs from a sphere, but the portion YZ corresponding to large angles of incidence lies within the sphere, the terminal point Y corresponding to light incident at a grazing angle at V and emergent parallel to the plane surface of the lens. In the lower part of the diagram a circle of the same radius as those through f and G has been drawn. Up to its intersection with the f circle at z no difference between the curves above and below the axis is seen. Beyond this point the portion yz is not symmetrical with YZ, but the difference is not large. This example shows again that the normal form for the secondary surface is a sphere, and that the departures from the sphere are small.

The extent of the aberrations present in such a lens may be exhibited by drawing curves indicating the secondary surfaces for different apertures—the harmonic sphere for any plano lens coincides with the spherical surface. It has already been shown by the aid of Fig. 13 that the aberrations in glass for parallel

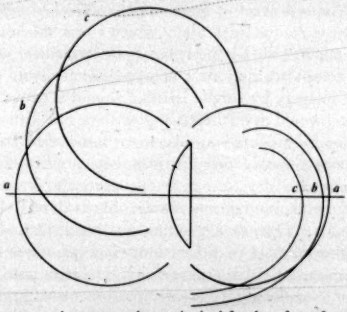

Fig. 23. Plano-convex lens: secondary principal focal surfaces for three apertures on the spherical surface

incident light are much smaller than those in air owing to the centres of the focal surfaces situated in the former medium being much nearer the centre of curvature of the refracting surface than those in the latter. In consequence of the translation of the surface from G to F brought about by cutting away the glass to the right of the plane W the centres of the surfaces are brought still nearer to C, and in consequence the aberration on the plane side of the lens undergoes a further diminution. This effect is seen in Fig. 23, where the focal surfaces marked a correspond to refraction at the axial point of the spherical surface, those marked c to a marginal point, and those designated b to a point midway between the axis and the margin. The small extent of the axial aberration on the plane side, particularly for moderate apertures, is in marked contrast with the large value on the spherical side of the lens. The curves b and c recall the form of a cardioid. The upper terminal points, where the curvature is greatest, correspond to light entering the lens through the spherical surface at grazing incidence, and the lower ends to light leaving the lens parallel to the plane surface. Study of this diagram will show why the curved

surface of such a lens should face a distant object when it is used as a telescope objective, and why the plane side should be nearer the eye or the image when it is used in an eyepiece.

It would be of interest to go farther and consider the characteristic grouping of secondary surfaces corresponding to the leading aberrations, spherical aberration, coma, pure astigmatism, and so on. To enter on this interesting part of the subject on this occasion would extend this address unreasonably, and I commend the problem to the members of the Society as one which they may find it entertaining to solve for themselves.

GEOMETRICAL OPTICS TREATED ALGEBRAICALLY

Rays and Waves

At this point I must leave this geometrical method of representing lens properties, for it is only one of the fields about which I wish to speak. If the discussion appears somewhat superficial, I hope it has been carried far enough to show that pencil, ruler, and compasses may be employed effectively to present at least the broader features of imagery by simple lenses. Should it prove capable of no more than this, the method would nevertheless appear to be not without value to students of physics, who may be expected to take more interest in the properties of real lenses than in the conventional, and quite impossible, optics of our text-books. To lens designers a comprehensive view of the properties of lenses of various shapes, such as this method yields, is obviously not devoid of interest. But we must beware of expecting too much. This, or any other, graphical method can do little more than enable a designer to start in a direction which promises to lead towards the point he wishes to reach. Effective progress along the right path can only be assured by the possession of, and the ability to use, instruments of greater precision.

At the present time there is no doubt that, of all the precise tools available, the one which has been used with the greatest measure of success by most of our lens designers is the method of ray tracing. Educationally the method appears to be of little value, and its successful application depends upon extensive experience. The information gained by its means has not yet been systematised and published; in fact the view has been expressed that this experience cannot be communicated by one person to another. Whether or not we accept that view, we may note that experienced designers, notwithstanding the very notable successes they have achieved by this method, would themselves welcome the appearance of other aids, either to replace ray tracing in its entirety, or to effect a marked reduction in the time devoted to calculations of this type*.

The current methods of manufacture and of design rest upon the assumption that only surfaces which are sensibly plane or spherical need be considered. On account of the very high accuracy necessary in these surfaces this limitation has been unavoidable for most optical work, but Mr Twyman has now provided the optician with a tool which will enable him, should he so desire, to reproduce any

* See a Discussion on "The Future of Geometrical Optics," *Trans. Opt. Soc.* **22** 1921 (201).

given surface to the highest accuracy called for in any optical instrument. Should investigation show that marked advantages are to be gained by departing from the spherical form, a profound alteration will be needed, not only in the processes of grinding and polishing, but also in the methods by which lenses are designed. The Twyman interferometer for testing and correcting lenses is likely to render another service within the optical industry of still greater importance. The technical optics of the present day is in effect built up on the reality of light rays. This conception, natural enough in the early days of the science, is known to be both incorrect and, in relation to optical instruments, generally misleading. Almost invariably the optician finds it necessary to effect compromises in correcting his lenses for aberrations. The fact that the geometrical theory of image formation by rays suggests a kind of compromise differing fundamentally from that to which the wave theory leads is thus of great moment to the designer. The interests of education and manufacture alike then compel us to consider the foundations of instrumental theory afresh from a new point of view. We shall find all that is needed in the famous, but utterly neglected, work of Sir William Rowan Hamilton, given to the world a hundred years ago.

The reality of light waves

Perhaps it will be as well, since recent discussions have tended to throw some doubt on the security of the wave theory of light, to turn aside for a moment and examine the sufficiency of the wave theory as a foundation for our studies. After the corpuscular theory had been definitely rejected, and a wave theory shown to be necessary, further discoveries for a time tended only to establish more securely the adequacy of the electromagnetic theory of light to account for all the observed phenomena. At a later period difficulties arose, first from mathematical, and subsequently from experimental, investigations. These have led to the belief that the energy of light, or at least all of it that we are able to measure, is radiated in compact finite quantities, instead, as was formerly supposed, of residing in a wave-motion susceptible of indefinite attenuation as it travels on through space. The suggestion was then put forward that, light being undoubtedly corpuscular and indivisible, waves were not needed, and probably did not exist.

A return to the older kind of theory was however impossible, and attempts were made to combine the essential properties of both theories by postulating the co-existence of quanta of energy travelling with the speed of light and of waves moving with the same velocity which, in the presence of matter, would set up fields of force tending to modify the directions in which the quanta moved. Other theories differing somewhat in form have been proposed, but in every case yielding the same effect. The conclusion to which they lead is that the quantum character of the light energy must be taken into account when we are studying isolated reactions between light and matter, such as those which determine the wave-length of the radiation, but that when we are dealing with properties in the mass, and particularly when we are concerned with the distribution of large quantities of energy in space, we must take the wave theory as our guide. For instance, the

form of an optical image, that is the statistical distribution of quanta in space over—on the light scale—an exceedingly long period of time, is determined correctly by the wave theory.

It may, however, be objected that since we see, and obtain all our other records, through reactions which essentially involve only the quantised energy, a theory which assumes the presence of waves contains something unnecessary, for which, in the very nature of the case, we have no evidence. For our purposes it is sufficient to reply that the distribution of light observed experimentally is in agreement with that calculated on the wave theory, and no alternative theory from which waves are excluded has been found to give the same result. The objection, however, can hardly be maintained when the action of a Michelson interferometer (see Fig. 24) is considered. The quantised energy leaving the lens L is by hypothesis indivisible, and can only take one or other of the alternative paths HM, HN after meeting the half-silvered mirror H. Since interference fringes are obtained after the light has been reflected back to this mirror, we are forced, in the absence of any alternative explanation, to assume that waves have travelled simultaneously by both paths. The portion of the wave which has been robbed of its quantum must be supposed, if we are to secure a simple explanation of the observations, to proceed with the same velocity as, and to vibrate with the frequency characteristic of, the quantum-bearing portion. In fact, if any distinction is to be drawn, the periodic effects we observe must be ascribed to the wave rather than to the quantum.

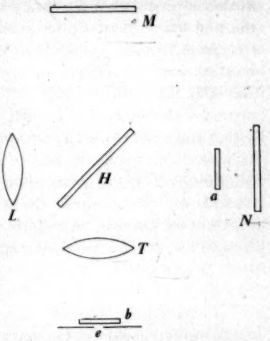

Fig. 24. Michelson interferometer

Since we can only become conscious of light in virtue of the quanta travelling with the waves, it might be thought that we could gain no further information about waves not so accompanied. Curiously enough this is not the case. We may show that these lone waves suffer absorption by materials which absorb the quanta. The general effects observed indeed are consistent with the view that the quanta travel through space because the waves carry them, and that when the waves are stopped by an absorbing medium the quanta stop and are absorbed because there is no wave to carry them forward. The demonstration of the absorption of lone wave trains may be effected by passing light of two distinct frequencies through a Michelson interferometer. Two filters, a and b, of reasonably good optical quality are required, one of which is transparent to the light of the greater of the two frequencies and opaque to that of longer wave-length, while the second is opaque to the former and transparent to the latter. The interferometer is set up with one of the filters a placed to intercept a part of the light travelling between the half-silvered mirror H and the reflector N, and adjusted to show a series of

broad interference fringes over the face of the filter. Such fringes will obviously be present since the filter is transparent to light of one of the frequencies employed. Fringes in both colours will be seen over that part of the field of view not covered by the filter. That there are fringes of the one colour only over the face of the filter may be proved by placing the second filter *b* in front of the observer's eye *e* in the re-united beams, when fringes of one colour only will be seen surrounding the first filter, the face of the filter itself appearing much less bright and entirely free from fringes. This result is precisely what would be anticipated on a pure wave theory. That it involves the absorption by the first filter *a* of the waves corresponding to the quanta it is known to stop is seen as follows. When a train of waves with its accompanying quantum reaches the half-silvered mirror the quantum must take one course or the other, but the waves divide, a train travelling upon each path. If the quantum travels along a path intersecting the filter *a*, it will be absorbed, and the whole wave train is of no further interest to us, since the only means by which our consciousness of its presence can be gained has been removed. If, on the other hand, the quantum pursues the alternative path *HM*, a lone wave train travels to the filter *a*, and, if not absorbed there, will ultimately return to the half-silvered mirror *H* and be capable of interfering with the quantum-bearing train reflected back from *M*. Thus if the wave train itself, though bearing no quantum, were not absorbed, interference fringes about half as bright as those surrounding the filter would be seen over the face of the filter *a* itself. The experiment illustrates that for these problems we may neglect the quantum aspect of light. The observed agreement of the distribution of light intensities with calculations on the older theory evidently implies that the probability of the quantum being found at a particular point of a wave is proportional to the energy of the wave motion in that neighbourhood.

Current views on Hamilton's method

We have now cleared the way for the discussion of the second of our uncultivated fields, the Hamiltonian treatment of optical instruments, the neglect of which by technical workers seems so unaccountable to those who regard the subject primarily as a branch of applied mathematics. Of all the methods which have been evolved for the discussion of optical problems, that of Hamilton is by far the most acceptable to the mathematical mind. At the present day it should make a special appeal to all who, accustomed to think in terms of rays, are trying to adjust their experience to the point of view of the wave theory, for the transfer from the one view to the other is in this system delightfully simple.

It would, however, be misleading to conceal that to some minds the method has appeared by no means simple. Indeed it would be easy to form the impression from the statements of well-known optical writers that it is beset by almost insuperable difficulties. The following extracts represent views which are widely held.

One of our Past Presidents, the late Professor Silvanus Thompson, writes:

> Remarkable as these researches of von Seidel are, it is of interest to note that an even more general method of investigation into lens aberrations had been previously propounded.

This is the fragmentary paper of Sir W. Rowan Hamilton, introducing into optics the idea of a "characteristic function," namely, the time taken by the light to pass from one point to another of its path. True, he did not work out the relations between the constants of his formulae and the data of the optical system. Yet the method, as a mathematical method of investigation, is unquestionably more powerful. It has recently, and independently, been revived by Thiesen, whose equations include those of von Seidel.

The latest development of advanced geometrical optics is due to Professor H. Bruns, who has shown that in general the formulae that govern the formation of images can be deduced from an originating function of the co-ordinates of the rays—a function termed by him the eikonal—by differentiating the same, just as in theoretical mechanics the components of the forces can be deduced by differentiation from the potential function. Brun's work is based upon the theory of contact-transformations of Sophus Lie. But as yet neither the formulae of Bruns nor those of Thiesen have been reduced to such shape as to be available for service in the numerical computation of optical systems*.

In a similar strain Professor Southall says:

If the characteristic function of a system is known, it is possible in theory to deduce from it all the optical properties of the system. In some comparatively simple cases this process enables us to get results with almost magical facility. It must be admitted, however, that the method, so fascinating on account of its generality, is difficult in its applications, involving as it does the theories of the higher analytical geometry and demanding mathematical knowledge and skill of the highest order. In addition to Hamilton, a number of other investigators, among whom may be mentioned especially Maxwell and Thiesen and Bruns, have developed in one way or another the theory of the characteristic function in optics. But the greatest difficulty is encountered in turning the theory to account, and, so far as the practical optician is concerned, the Hamiltonian method has not been found to smooth his way†.

That these views are not singular to English-speaking authorities will be clear from the view of Dr H. Siedentopf:

When the characteristic function is known for any optical system all the properties of that system may be deduced theoretically from it. These theoretical deductions involve, however, the use of an elaborate analytical equipment, particular use being made of the theory of curved surfaces and that of pencils. Whilst it will thus be seen that the efforts of these few investigators have already gone far to erect upon the theory of these general systems an intricate analytical structure, which is only accessible to those endowed with very extensive mathematical resources, the difficulties to which these theories lead become insuperable as we proceed to their practical application. Hitherto the practical embodiment of the characteristic function has proved realisable in the very simplest cases only. Unfortunately, cases of this kind either have no practical significance or have already been solved for practical purposes by simple specialised methods. Scarcely less are the difficulties which are encountered if one proceeds to secure practical working formulae by applying Fermat's principle as a means of eliminating the intermediate variables associated with the points where the rays intersect the successive surfaces. Moreover, the extremely general nature of such analytical investigations does not at all provide the practical optician with that which interests him most, viz. standard forms for the computation of optical combinations‡.

* Thompson, *Photographic Optics*, p. 114.

† Southall, *Principles and Methods of Geometrical Optics*, pp. 38–39, § 41.

‡ von Rohr, *Principles of Geometrical Optics*, p. 23.

Notwithstanding this great weight of adverse opinion I hope to show that the difficulties of the method are less formidable than they have appeared to these workers, and that, provided we are ready to conform to the natural bias of the method itself, and refrain from the attempt to force it into channels more appropriate to other methods of investigation, there is nothing to prevent its proving an extremely efficient tool in the hands of any reasonably well equipped investigator.

The validity of algebraic expansions

Before entering on the details of this algebraic method of dealing with optical instruments, we may turn aside to consider an ingenious objection raised by Captain Baker to a class of algebraic expansions which have been used to express their aberrations. Fig. 25 illustrates an optical system in which a beam of light travelling from an infinitely distant point on the axis passes through the lens from right to left, is thereby converged, and, owing to the presence of aberration, touches a caustic surface. F represents the principal focus for paraxial rays, FG is part of the caustic surface, and one of the incident rays hH' after refraction meets the axis

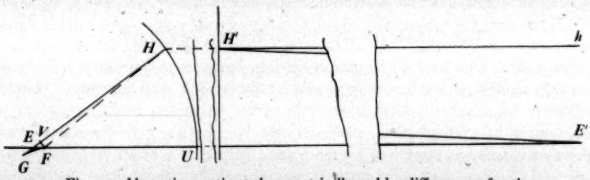

Fig. 25. Aberrations estimated geometrically and by differences of path

at E and the caustic surface at G. If now we consider E as the origin of a beam of light travelling in the reverse direction and refracted by the lens, the paraxial rays will converge to a point E' at a great but finite distance from the lens. The ray EH will emerge parallel to the axis as $H'h$, meeting the axis at infinity. The longitudinal aberration for this ray is therefore infinitely great, and any expression for the aberration which gives a finite value, no matter how great, makes an infinitely great error. This reasoning leads Captain Baker to reject expressions of the usual form and to question the validity of series expansions. The only proper algebraic form for the aberrations, he concludes, is a fraction, and in collaboration with Professor Filon he has succeeded in constructing algebraic fractions which represent the longitudinal spherical aberrations of single lenses with very considerable accuracy*.

That these considerations have led to the construction of new formulae of such accuracy is all to the good, but there the matter seems to end. Expressions open to this objection arise only through unnatural treatment, usually the adoption of unsuitable variables. The objections do not hold for the expansions to be employed in the development of Hamilton's method. But even where the objection would

* *Trans. Opt. Soc.* **20** 1918 (67); *Phil. Trans.* A **221** 1920 (29).

apply the difficulty can easily be evaded, for there is no need to measure aberrations longitudinally, and the lateral aberration is always finite. The real answer to the objection however rests upon other grounds. The argument assumes that geometrical reasoning leads to correct ideas of optical imagery and of the importance of aberrations. This is far from being the case. This aberration, which we find infinitely great from geometrical reasoning, may physically be indistinguishable from zero aberration, for the outstanding path differences may only amount to a small fraction of a wave-length of light. If H is on the unit surface and $HV = HF$, the path difference at E' depends on the path difference at F for the light originally considered, together with the difference between EV and EF, and also the difference between $H'E'$ and the axial distance of E' from the normal plane through H'. The last mentioned difference is trivial, and the path difference at E' in the case we are invited to consider is effectively that at E for the initial parallel beam of light. The only significant measure of the aberration therefore is not an infinitely great quantity, but a very small one. Another method of stating this conclusion is to say that the diffraction pattern at E' is of dimensions comparable with hE', and the effect of this lateral aberration is therefore slight.

The Characteristic Function and the Eikonal

The reader who goes to Hamilton's original papers to gain knowledge which may help in optical studies must needs be forearmed with patience. Hamilton apparently takes not the least interest in optics or lenses, but aims only at the development of mathematical methods. He is ever ready to forsake the main highway and follow a bypath as his fancy may dictate. Those attracted to the work by reason of its mathematical interest may cheerfully follow along this rambling path; but the student seeking only weapons wherewith to attack his optical problems must expend much labour for all his gains. To these it may be of value to outline a much more direct course, in which no attempt is made to adhere to Hamilton's own treatment. In his published work entitled *Systems of Rays*, comprising an initial paper and three supplements, Hamilton makes detailed use of three distinct functions, each of which specifies implicitly all the properties of an optical instrument. He speaks of all these as "characteristic functions" of the instrument, but when mention is made of the characteristic function in modern books the particular function which occurs earliest in these papers is generally to be understood. This function represents the length of the optical path between two points separated by the instrument, expressed as a function of their cartesian coordinates relative to assigned systems of reference axes. It is admirably suited for theoretical research, and would undoubtedly be useful in technical applications but for the difficulty of forming the function for a particular instrument. Owing to this difficulty the balance of advantage seems to lie with one of the functions introduced at a later stage of Hamilton's investigations, in which point coordinates have been replaced by line coordinates. We shall adhere to the restricted use of characteristic function for the first of these functions, and for the latter, the properties of which we are about to investigate, we will use Bruns' descriptive term, the eikonal.

Light from a given source travels out in all directions at a uniform velocity provided (as is here always assumed to be the case) the medium concerned is homogeneous. At any given instant it reaches positions which lie on a sphere with the origin as centre. The rays are defined as the normals to this sphere. When refraction takes place the light, on entering the new medium, travels with a different velocity, and the boundary of the space which the light has traversed at any instant will in general no longer be spherical. At every moment the boundary is continuous, and in any given medium expands along the normals everywhere at the same rate, so that the boundary varies continuously in direction. These normals or rays are, within small limits, the paths along which the radiant energy is transferred, and from Huygens' principle they obey the sine law of refraction. We first show how the rays may be identified by differentiating a function representing a certain length measured along the ray.

Suppose in the first place that a point source of light is situated at (x, y, z) in the object space. Light is radiated from this point in all directions, and some traverses the optical system and finds its way into the image space. Generally the wave-front in the image space will be a curved surface, and the normals at different points of this surface differ in direction. A particular emergent ray may therefore be specified by its direction cosines. Let these be (L', M', N'), and suppose also that (ξ', η', ζ') is a point on this ray. The disturbance has taken a definite time to reach (ξ', η', ζ') from (x, y, z): let the corresponding optical path length, that is the distance light travels in a standard medium in this time, be denoted by $\mathscr{U}$. Let $\mathscr{U} + \delta\mathscr{U}$ be the path length for a neighbouring ray starting from (x, y, z) and finishing at $(\xi' + \delta\xi', \eta' + \delta\eta', \zeta' + \delta\zeta')$, the final direction being $(L' + \delta L', M' + \delta M', N' + \delta N')$. If μ' is the refractive index of the final medium, the second path exceeds the first by

$$\mu'\{(L' + \delta L')\,\delta\xi' + (M' + \delta M')\,\delta\eta' + (N' + \delta N')\,\delta\zeta'\} = \delta\mathscr{U},$$

since the wave-front, which marks the locus of points optically equidistant from (x, y, z), is normal to the ray. Now suppose that the tangent planes to the wave-fronts at (ξ', η', ζ') and $(\xi' + \delta\xi', \eta' + \delta\eta', \zeta' + \delta\zeta')$ pass through the origin. Then

$$\xi'L' + \eta'M' + \zeta'N' = 0,$$

$$(\xi' + \delta\xi')\,(L' + \delta L') + (\eta' + \delta\eta')\,(M' + \delta M') + (\zeta' + \delta\zeta')\,(N' + \delta N') = 0,$$

and these conditions enable the expression we have found for the path difference to be written

$$\delta\mathscr{U} = -\mu'\,(\xi'\delta L' + \eta'\delta M' + \zeta'\delta N') \qquad \ldots\ldots(1).$$

Now L', M', N' are connected by the relation

$$L'^2 + M'^2 + N'^2 = 1,$$

and therefore only two of them, say M' and N', may be regarded as independent variables. On eliminating $\delta L'$, (1) becomes

$$\delta\mathscr{U} = -\mu'\left\{\left(\eta' - \xi'\frac{M'}{L'}\right)\delta M' + \left(\zeta' - \xi'\frac{N'}{L'}\right)\delta N'\right\}.$$

Since $\mathscr{U}$ is now regarded as a function of M' and N', for any small variations of M' and N' we shall have

$$\delta\mathscr{U} = \frac{\partial\mathscr{U}}{\partial M'}\delta M' + \frac{\partial\mathscr{U}}{\partial N'}\delta N',$$

and, since M' and N' may be varied independently, we find, by comparing these equations,

$$\frac{\partial\mathscr{U}}{\partial M'} = -\mu'\left(\eta' - \xi'\frac{M'}{L'}\right),$$

$$\frac{\partial\mathscr{U}}{\partial N'} = -\mu'\left(\zeta' - \xi'\frac{N'}{L'}\right).$$

Now $\eta' - \xi'\dfrac{M'}{L'}$ and $\zeta' - \xi'\dfrac{N'}{L'}$ are invariants for any given ray, and represent the y' and z' coordinates of the intersection of the ray with the plane $x' = 0$. If we understand that y' and z' are the coordinates of a point in this plane we may write these equations

$$\left.\begin{aligned}\mu' y' &= -\frac{\partial\mathscr{U}}{\partial M'}\\ \mu' z' &= -\frac{\partial\mathscr{U}}{\partial N'}\end{aligned}\right\} \qquad \text{.....(2).}$$

If then we know $\mathscr{U}$ as a function of M' and N', that is to say if we know how the length of the optical path from the source to a plane through the image space origin varies as the direction of this plane alters, we can find where the common normal to this plane and the wave meets an arbitrary fixed plane through the origin.

If, instead of starting with a source of light at a known point of the object space, we had assumed it to be situated in the image space, some of the light diverging from this point would reach the lens and after refraction would emerge into the object space. We could take $\mathscr{U}'$ as a measure of the time taken by the light to reach a plane in the direction (L, M, N) passing through the origin, and obtain the equations

$$\left.\begin{aligned}\mu y &= \frac{\partial\mathscr{U}'}{\partial M}\\ \mu z &= \frac{\partial\mathscr{U}'}{\partial N}\end{aligned}\right\} \qquad \text{.....(3)}$$

for the point (y, z) in which the common normal to the wave-front and the plane $L\xi + M\eta + N\zeta = 0$ met the plane $x = 0$. The change of sign which will be observed on comparing equations (2) and (3) is due to the assumption that the positive directions of the axes are unaltered, so that a positive displacement of (ξ, η, ζ) corresponds to a *decrease* in the time $\mathscr{U}'$.

The function $\mathscr{U}$ suffers from the grave disadvantage that it is unsymmetrical, the variables in the object space being point coordinates, and in the image space direction cosines. A function which is symmetrical is at once obtained by considering the particular case in which the origin of the light is at infinity. It is

inconvenient to include in the function the infinite term representing the length of the path between the source and a reference position near the lens, so the path is measured from the wave-front which passes through the object space origin. Since all points on the same wave-front are at the same optical distance from the source, $\mathscr{E}$, the new finite path length, which is a function of (M, N, M', N'), differs from $\mathscr{U}$ by a constant, and, as in the case of $\mathscr{U}$, the equations

$$\left.\begin{aligned} \mu' y' &= -\frac{\partial \mathscr{E}}{\partial M'} \\ \mu' z' &= -\frac{\partial \mathscr{E}}{\partial N'} \end{aligned}\right\} \qquad \text{.....(4)}$$

are satisfied. In a similar way from $\mathscr{U}'$, by placing the source at infinity in the image space in the direction L', M', N' and rejecting the constant infinite part of the path, we obtain a finite function $\mathscr{E}'$ having (M, N, M', N') as its variables and satisfying the equations

$$\left.\begin{aligned} \mu y &= \frac{\partial \mathscr{E}'}{\partial M} \\ \mu z &= \frac{\partial \mathscr{E}'}{\partial N} \end{aligned}\right\} \qquad \text{.....(5).}$$

Now $\mathscr{E}$ and $\mathscr{E}'$ measure the time taken by the light to travel between the same two planes in opposite directions along the stationary path. The stationary path between two planes, it is to be observed, is unique, being independent of the curvature of the wave-front at either plane. Moreover the speed of light is independent of direction. It follows that $\mathscr{E}$ and $\mathscr{E}'$ are equal, and, since they are expressed in terms of the same variables, they must be identical. It is this symmetrical function of the direction cosines which we name the eikonal.

In the application of this function the planes $x = 0$ and $x' = 0$ will be chosen

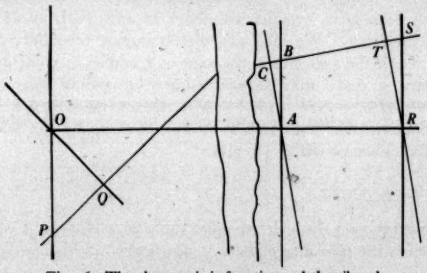

Fig. 26. The characteristic function and the eikonal

to coincide with the object and image planes, or at least to be parallel to them. Equations (4) and (5) then show how this function, which itself expresses the length of a path carried through the refracting surfaces, and is thus particularly

suitable for investigations according to the wave theory, enables the points in which rays travelling in specified directions meet the object and image surfaces, that is to say the information which a designer is accustomed to derive by ray tracing, to be found. It is of special importance to bear in mind that the eikonal is not the path in which we are ultimately interested, though closely connected with it. In Fig. 26, which for simplicity represents a path in a plane, OP represents the object plane, RS the image plane. We are ultimately interested in the equality of all optical paths between the point P and its image S. The eikonal $\mathscr{E}$ on the other hand represents the length of the path from Q to T, between planes normal to the ray through O and R. This does not mean that further calculation is necessary to find the state of correction of the system. It may be observed that there is a difficulty at least as great when the characteristic function is used. The conjugate end points P and S do not specify any path uniquely, and if their coordinates are inserted the characteristic function assumes an indeterminate form. It is necessary to select for at least one of the points whose coordinates appear in the characteristic function a point in a plane remote from the object and image surfaces. For instance, the point B in the plane AB may be used instead of S. If we desire we can of course use A instead of R as a reference point in constructing the eikonal, in which case the eikonal will represent the length of the path between Q and C.

Focal lengths and principal foci

The computer usually employs ray tracings not so much as a means of finding from each alone the properties of the system he is investigating, as to derive the positions of foci by comparing the paths of a number of rays. Corresponding information can be derived from the eikonal with great ease. The directions of the rays are determined by the law that the path between two points is stationary for small variations of path, and the foci are points on a particular path between which the paths are stationary for larger displacements. We shall restrict ourselves in what follows to systems symmetrical about an axis, such as we have already considered geometrically. We then saw that conjugate secondary foci lie in the same plane through the axis. Take the axes of x and x' in coincidence with the axis of symmetry, so that $\mathscr{E}$ may be regarded as a function of three variables only, viz. $\frac{1}{2}(M^2+N^2)$, $MM'+NN'$, $\frac{1}{2}(M'^2+N'^2)$. Denote these by a, b, c respectively, and denote differentiation with respect to any of them by the addition of this letter as a suffix. Then equations (4) give

$$\mu' y' = -M\mathscr{E}_b - M'\mathscr{E}_c,$$
$$\mu' z' = -N\mathscr{E}_b - N'\mathscr{E}_c,$$

that is to say the ray goes through the point $(\rho' L', y'+\rho' M', z'+\rho' N')$ distant ρ' along the ray from the reference plane $x'=0$, where

$$\rho' = \frac{1}{\mu'}\mathscr{E}_c \qquad \text{.....(6),}$$

provided

$$\mu'(y'+\rho' M') = -M\mathscr{E}_b,$$
$$\mu'(z'+\rho' N') = -N\mathscr{E}_b.$$

These two equations show that at the distance ρ' from the reference plane the ray goes through a point in the axial plane containing the infinitely distant origin of light (L, M, N). In other words the secondary principal focus lies on the ray at a distance $(1/\mu')\,\mathscr{E}_c$ beyond the point in which it meets the reference plane $x' = 0$.

In a similar way the secondary principal focus in the object space lies at a distance $-(1/\mu)\,\mathscr{E}_a$ from the intersection of the ray with the reference plane $x = 0$, the measurement being made in the positive direction.

If f and f' are the secondary focal lengths of the system for this ray in the object and image spaces, the y' and z' coordinates of the secondary image of the infinitely distant object (L, M, N) are Mf' and Nf' respectively. It follows from the last two equations that

$$\left.\begin{aligned} \mu' f' &= -\mathscr{E}_b \\ \text{and similarly} \quad \mu f &= -\mathscr{E}_b \end{aligned}\right\} \qquad \ldots\ldots(7).$$

Thus simple differentiation with respect to a, b, c gives in the most convenient form information usually obtained indirectly, and much more laboriously, by ray tracing.

An alternative method of stating (7) is to say that the focal lengths are μ/p and μ'/p respectively where $p = -\mu\mu'/\mathscr{E}_b$. It will be observed that p is the secondary power of the system for this ray*.

These results may be illustrated by taking for the system a single spherical surface of radius r. Let the two sets of reference axes coincide, and take the centre of curvature as origin. Then

$$\mathscr{E} = r\,(\mu' \cos\phi' - \mu \cos\phi),$$

and, since $\mu' \sin\phi' = \mu \sin\phi$,

$$\begin{aligned} (\mu' \cos\phi' - \mu \cos\phi)^2 &= \mu'^2 + \mu^2 - 2\mu'\mu \cos(\phi' - \phi) \\ &= \mu'^2 + \mu^2 - 2\mu'\mu\,(LL' + MM' + NN') \\ &= \mu'^2 + \mu^2 - 2\mu'\mu\,(1 - 2a)^{\frac{1}{2}}(1 - 2c)^{\frac{1}{2}} - 2\mu'\mu b, \end{aligned}$$

so that
$$\begin{aligned} \mathscr{E}_b &= -\mu'\mu r\,\{\mu'^2 + \mu^2 - 2\mu'\mu\,(1 - 2a)^{\frac{1}{2}}(1 - 2c)^{\frac{1}{2}} - 2\mu'\mu b\}^{-\frac{1}{2}} \\ &= -\frac{\mu\mu' r}{\mu' \cos\phi' - \mu \cos\phi}, \end{aligned}$$

and
$$\begin{aligned} \mathscr{E}_c &= \frac{\mu'\mu\,(1 - 2a)^{\frac{1}{2}} r}{(1 - 2c)^{\frac{1}{2}}}\,\{\mu'^2 + \mu^2 - 2\mu'\mu\,(1 - 2a)^{\frac{1}{2}}(1 - 2c)^{\frac{1}{2}} - 2\mu'\mu b\}^{-\frac{1}{2}} \\ &= \frac{L}{L'} \cdot \frac{\mu\mu' r}{\mu' \cos\phi' - \mu \cos\phi}. \end{aligned}$$

Now referring to Fig. 27, the principal focus F' on the refracted ray lies also on the parallel through the centre of curvature C to the incident ray. Let the angles of incidence and refraction be ϕ and ϕ' and those made by the rays with the axis ψ

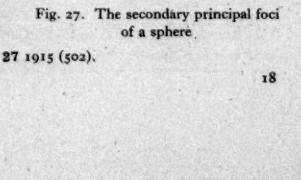

Fig. 27. The secondary principal foci of a sphere

* *Proc. Phys. Soc.* **27** 1915 (502).

and ψ'. Then in the triangle PCF' the angles are, in order, ϕ', $\pi - \phi$, and $\phi - \phi'$, so that

$$\frac{PF'}{\sin\phi} = \frac{CF'}{\sin\phi'} = \frac{r}{\sin(\phi - \phi')},$$

or

$$\mu PF' = \mu' CF' = \frac{\mu\mu' r}{\mu'\cos\phi' - \mu\cos\phi}.$$

It follows that $\mu PF' = -\mathscr{E}_b$ in accordance with (7). Similarly we find $\mu' FP = -\mathscr{E}_b$. An alternative proof may be obtained by starting from the known value $\frac{\mu'\cos\phi' - \mu\cos\phi}{r}$ of the secondary power.

Now let the rays meet the plane normal to the axis through C in B and D. Projecting on to the axis

$$DF'\cos\psi' = CF'\cos\psi,$$

or

$$DF' = \frac{1}{\mu'}\cdot\frac{L}{L'}\cdot\frac{\mu\mu' r}{\mu'\cos\phi' - \mu\cos\phi}$$

$$= \frac{1}{\mu'}\mathscr{E}_c.$$

In a similar way we may obtain $FB = \frac{1}{\mu}\mathscr{E}_a$.

That the positions of the origins may be selected arbitrarily is seen by putting $\mathscr{E}'$ for the eikonal when they are moved to $(x, 0, 0)$ and $(x', 0, 0)$ respectively referred to the origin at the centre of curvature. From the definition of the eikonal it follows that

$$\mathscr{E}' = \mathscr{E} - \mu x L + \mu' x' L'$$

and thus by differentiation

$$\mathscr{E}_a' = \mathscr{E}_a + \frac{\mu x}{L}, \quad \mathscr{E}_b' = \mathscr{E}_b, \quad \mathscr{E}_c' = \mathscr{E}_c - \frac{\mu' x'}{L'},$$

showing that the differentiation of the new function gives the same value for the focal length and the same points as principal foci.

We now turn to the corresponding primary quantities. The coordinates Y', Z' of a point on the emergent ray distant ρ' from the intersection of the ray with the plane $x' = 0$ have been shown to satisfy

$$\left.\begin{aligned}\mu' Y' &= (\mu'\rho' - \mathscr{E}_c)M' - \mathscr{E}_b M\\ \mu' Z' &= (\mu'\rho' - \mathscr{E}_c)N' - \mathscr{E}_b N\end{aligned}\right\} \qquad \text{.....(8).}$$

For a neighbouring ray incident in the same direction and having the same value of X', i.e. $\rho' L'$, and for which the variables are therefore $M, N, M' + \delta M', N' + \delta N'$, we find corresponding coordinates $Y' + \delta Y', Z' + \delta Z'$, where

$$\mu'\delta Y' = (\mu'\rho' - \mathscr{E}_c)\delta M' + \left(\frac{\mu' M'\rho'}{L'^2} - M\mathscr{E}_{bc} - M'\mathscr{E}_{cc}\right)(M'\delta M' + N'\delta N')$$
$$- (M\mathscr{E}_{bb} + M'\mathscr{E}_{bc})(M\delta M' + N\delta N'),$$

$$\mu'\delta Z' = (\mu'\rho' - \mathscr{E}_c)\delta N' + \left(\frac{\mu' N'\rho'}{L'^2} - N\mathscr{E}_{bc} - N'\mathscr{E}_{cc}\right)(M'\delta M' + N'\delta N')$$
$$- (N\mathscr{E}_{bb} + N'\mathscr{E}_{bc})(M\delta M' + N\delta N').$$

From these values we find

$$\mu'(M\delta Y' + N\delta Z')$$
$$= (M\delta M' + N\delta N')\left(\frac{\mu'\rho'}{L'^2} - \mathscr{E}_c - 2a\mathscr{E}_{bb} - 2b\mathscr{E}_{bc} - 2c\mathscr{E}_{cc}\right)$$
$$+ (M'N - MN')\left\{\mathscr{E}_{bc}(M\delta N' - N\delta M') - \left(\frac{\mu'\rho'}{L'^2} - \mathscr{E}_{cc}\right)(M'\delta N' - N'\delta M')\right\}.$$

Now for rays in a primary plane $M'N - MN' = 0$, and the primary focus is given by the conditions $\delta Y' = \delta Z' = 0$. It follows that the distance ρ' to the primary focus is given by

$$\rho' = \frac{L'^2}{\mu'}(\mathscr{E}_c + 2a\mathscr{E}_{bb} + 2b\mathscr{E}_{bc} + 2c\mathscr{E}_{cc})$$

and similarly for the object space

$$\rho = -\frac{L^2}{\mu}(\mathscr{E}_a + 2a\mathscr{E}_{aa} + 2b\mathscr{E}_{ab} + 2c\mathscr{E}_{ac}) \qquad \text{.....(9).}$$

The primary focal length F' in the image space is defined by the equation

$$F' = \frac{\delta n'}{\delta\psi},$$

where $\delta n'$ is the displacement of the focus in the primary plane perpendicular to the ray consequent on an increase $\delta\psi$ in the angle made by the incident ray with the axis. Since $\delta \sin\psi = \cos\psi\delta\psi$, we may replace $\delta\psi$ by $\delta M/L$, where we assume refraction to take place in the plane $Z = Z' = 0$, so that $N = N' = 0$. $\delta n'$ is evidently obtained by putting $\delta\rho' = 0$, $L'\delta Y' = \delta n'$, so that $F' = LL'(\delta Y'/\delta M)$. Now varying M, that is to say a and b in (8), we obtain under these conditions

$$\mu'\delta Y' = -(\mathscr{E}_b + M^2\mathscr{E}_{ab} + MM'\mathscr{E}_{bb} + MM'\mathscr{E}_{ac} + M'^2\mathscr{E}_{bc})\,\delta M,$$

which evidently corresponds to the more symmetrical form

$$\mu'F' = -LL'(\mathscr{E}_b + 2a\mathscr{E}_{ab} + b\mathscr{E}_{bb} + b\mathscr{E}_{ac} + 2c\mathscr{E}_{bc}).$$

Similarly we obtain

$$\mu F = -LL'(\mathscr{E}_b + 2a\mathscr{E}_{ab} + b\mathscr{E}_{bb} + b\mathscr{E}_{ac} + 2c\mathscr{E}_{bc}) \qquad \text{.....(10),}$$

for the primary focal length F in the object space. In terms of the primary power P, $F = \mu/P$, $F' = \mu'/P$, so that P is equal to $\mu\mu'$ divided by the expression on the right of (10).

We may now verify these expressions for a single spherical surface. Noting that we have already found that

$$\frac{L}{L'}\cdot\frac{\partial(\mu'\cos\phi' - \mu\cos\phi)}{\partial a} = -\frac{\partial(\mu'\cos\phi' - \mu\cos\phi)}{\partial b} = \frac{L'}{L}\cdot\frac{\partial(\mu'\cos\phi' - \mu\cos\phi)}{\partial c}$$
$$= \frac{\mu\mu'}{\mu'\cos\phi' - \mu\cos\phi},$$

and observing further that, since the rays are in a primary plane, $MN' - M'N = 0$ and therefore

$$L'^2(M^2 + N^2) - 2LL'(MM' + NN') + L^2(M'^2 + N'^2) = \sin^2(\psi - \psi')$$
$$= \sin^2(\phi - \phi'),$$

we are led, on carrying out the differentiations, to the expressions

$$\left.\begin{aligned} \mathscr{E}_b + 2a\mathscr{E}_{ab} + b\mathscr{E}_{bb} + b\mathscr{E}_{ac} + 2c\mathscr{E}_{bc} &= -\frac{\mu\mu' r\cos\phi\cos\phi'}{LL'(\mu'\cos\phi' - \mu\cos\phi)} \\ \text{and} \qquad \mathscr{E}_c - L'^2\{\mathscr{E}_c + 2a\mathscr{E}_{bb} + 2b\mathscr{E}_{bc} + 2c\mathscr{E}_{cc}\} &= \frac{\mu' r\sin\phi\sin^2\phi'}{\sin(\phi-\phi')} \end{aligned}\right\} \ldots(11).$$

The known value of the primary power* is

$$\frac{\mu'\cos\phi' - \mu\cos\phi}{r\cos\phi\cos\phi'},$$

thus verifying the expression for the focal lengths, and the accuracy of the other expression may be seen on referring to Fig. 28. CB and CD are perpendiculars to the rays from the centre of curvature, and BD meets the line joining C to the intersections I and J of the rays with the aplanatic surfaces perpendicularly at K. It at once follows that $CK = r\sin\phi\sin\phi'$. The point K is the centre of perspective for primary rays just as C is for the secondary rays†. The astigmatism in the image space is therefore obtained by projecting CK on the refracted ray in the direction of the incident ray, with a corresponding construction for the object space. Since the angles made by CK with the incident and emergent rays are ϕ' and ϕ respectively, the values of the astigmatism for the object and image spaces are respectively

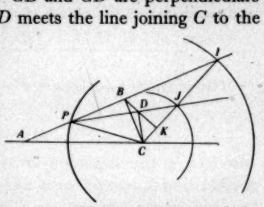

Fig. 28. The primary principal foci of a sphere

$$\frac{CK\sin\phi}{\sin(\phi-\phi')} \text{ and } \frac{CK\sin\phi'}{\sin(\phi-\phi')}.$$

By (6) and (9) the image space astigmatism is obtained by dividing the expression on the left by (11) of μ'; the verification is thus completed.

Various applications of the eikonal

In any optical instrument the quantity of physical significance is the variation in the length of the whole optical path between an object point and its image. As has been already pointed out $\mathscr{E}$ is not a measure of this quantity, since the terminal points of the path which it measures are the feet of the perpendiculars to the ray from the reference points on the axis. However, the length of path required is easily found. For y' and z', the displacements perpendicular to the axis from the axial reference points to the point of intersection of the ray with the normal plane, if projected on the ray, give the distance of the latter point from the terminal point for which $\mathscr{E}$ is evaluated, and if this distance is multiplied by the refractive index μ' the corresponding optical path is obtained. Thus in the image space an addition to $\mathscr{E}$ of $\mu'(M'y' + N'z')$ must be made. In the object space the corresponding quantity is $\mu(My + Nz)$, and this has to be subtracted. From the

* *Proc. Phys. Soc.* **30** 1918 (221). † Thomas Young, *A course of lectures on natural philosophy*.

equations giving y, z, y', z' by the differentiation of $\mathscr{E}$, it follows that the length of the path between the object and image planes is

$$\mathscr{E} - M\frac{\partial \mathscr{E}}{\partial M} - N\frac{\partial \mathscr{E}}{\partial N} - M'\frac{\partial \mathscr{E}}{\partial M'} - N'\frac{\partial \mathscr{E}}{\partial N'}.$$

Since a, b, c are homogeneous functions of M, N, M', N' of the second degree this may be written

$$\mathscr{E} - 2(a\mathscr{E}_a + b\mathscr{E}_b + c\mathscr{E}_c),$$

and in the length of the path the aberrations of order n, that is the terms of order $n+1$ in a, b, c, have $-(2n+1)$ times the weight which they carry in $\mathscr{E}$. The minus sign indicates that the physical path involved and $\mathscr{E}$ vary in opposite directions, an increase in path corresponding to a decrease in $\mathscr{E}$, and vice versa. For reasons which cannot be discussed here this process is not required, the alteration being compensated by the way in which the function is used.

It might at first sight appear that, in comparison with the characteristic function, the eikonal suffers from a disadvantage because it does not represent the length of the path between the planes which form the ideal conjugate surfaces. Closer consideration however shows that in reality the disadvantage may lie in the other direction. If it is merely sought to determine what conditions must be satisfied to ensure the absence of aberrations, any functions may be employed. The resulting equations may differ in form, particularly in the case of aberrations of high orders, but in fact any one set of conditions is algebraically reducible to any other set. To distinguish between the alternative methods it is necessary to consider the more general case when aberrations are present. From the physical point of view the length of path between the fundamental terminal planes measured along the ray is not the quantity required. The disturbance at any particular point of an intermediate wave-front travels in all directions, and not merely along the normals or ray paths, and the phase variation must be sought by finding the length of path along routes which are not normal to the wave-fronts. At least in the cases of practical importance, where the light distribution sought is in the immediate neighbourhood of what passes for the geometrical image, this phase variation may be found by assuming that the position of the intermediate wave-front where the divergence from the ray path begins is very remote. This leads to the expression

$$\text{Optical path} = \mathscr{E} - \mu(My + Nz) + \mu'(M'y' + N'z').$$

In form this equation agrees with the old expression, but the interpretation is different; y' and z' are no longer the coordinates of the point in which the ray strikes the image reference plane, but are the coordinates of the arbitrary point at which the light intensity is to be determined. Taking the case when the object is at infinity, and excluding the infinite part of the path in the object space, the quantity of physical importance is seen to be

$$\mathscr{E} + \mu'(M'y' + N'z').$$

In applying this expression it is best to replace y' and z' by $y_0' + \delta y'$ and $x_0' + \delta z'$, where y_0' and z_0' are the Gaussian image points; it then becomes

$$\mathscr{E} + \mu\mu' (MM' + NN')/L\kappa + \mu' (M'\delta y' + N'\delta z'),$$

where κ is the paraxial power. If the reference plane does not coincide with the paraxial focal plane the additional term

$$\mu' L' \delta x'$$

must be added, where $\delta x'$ is the x' coordinate of the actual reference plane with respect to the paraxial focal plane.

The illustrations hitherto considered relate to the very important case when the object is at an infinite distance from the lens. It must not however be thought that the eikonal may only be employed under these limited conditions. When near objects are considered it is simplest to change the variables, replacing a, b, and c by A, B, C*, where

$$\begin{aligned}(S-G)^2 A &= a - Gb + G^2 c,\\ (S-G)^2 B &= 2a - (S+G) b + 2SGc,\\ (S-G)^2 C &= a - Sb + S^2 c.\end{aligned}$$

This substitution is always permissible, for it is equivalent to

$$\begin{aligned}a &= S^2 A - SGB + G^2 C,\\ b &= 2SA - (S+G) B + 2GC,\\ c &= A - B + C.\end{aligned}$$

The condition for freedom from aberrations at magnification G is then that $\mathscr{E}$ is a function of A only. For adopting for the moment the familiar ray conception of imagery,

$$\begin{aligned}y &= \frac{\partial \mathscr{E}}{\partial M} = M\mathscr{E}_a + M'\mathscr{E}_b\\ &= \frac{1}{(S-G)^2}\{(M - GM')(\mathscr{E}_A + \mathscr{E}_B) + (M - SM')(\mathscr{E}_B + \mathscr{E}_C)\},\end{aligned}$$

$$y' = -\frac{\partial \mathscr{E}}{\partial M'} = \frac{1}{(S-G)^2}\{(M - GM')(G\mathscr{E}_A + S\mathscr{E}_B) + (M - SM')(G\mathscr{E}_B + S\mathscr{E}_C)\},$$

so that

$$y' - Gy = \frac{1}{S-G}\{(M - GM')\mathscr{E}_B + (M - SM')\mathscr{E}_C\},$$

and similarly

$$z' - Gz = \frac{1}{S-G}\{(N - GN')\mathscr{E}_B + (N - SN')\mathscr{E}_C\}.$$

Thus, if the reference points lie in the planes conjugate to one another for magnification G, the presence of terms in B and C with finite coefficients involves displacements of the image points from their desired positions. The aberrations are

* The equations have been simplified by assuming that the system is immersed in air.

therefore measured by the coefficients of the terms in $\mathscr{E}$ which involve the variables B and C.

This introduction to the properties of the eikonal will perhaps suffice to show how readily it yields the various types of information which the designer may require. Before passing on to another aspect of the problem we may stop to show by its aid that the constant magnification secondary surfaces of ideally corrected instruments, like those of the single surfaces we considered earlier, are spherical. We will assume a system having plane object and image surfaces entirely free from aberration for the magnification G, and in addition no aberration at the nodal points. This condition secures complete absence of distortion, curvature, and astigmatism with a stop placed at one of these points. The eikonal for the system is given by*

$$\mathscr{E} = -\frac{f}{G}(1-G)^2(1-2A)^{\frac{1}{2}},$$

where f is the paraxial focal length. The secondary focal length for a general ray, by equation (7), is

$$-\mathscr{E}_b = f(1-2A)^{-\frac{1}{2}},$$

and is therefore constant for all rays passing through a given point of the aberration-less object or image planes. The constant magnification surfaces for all rays passing through the given point and its image are therefore spheres with these points as

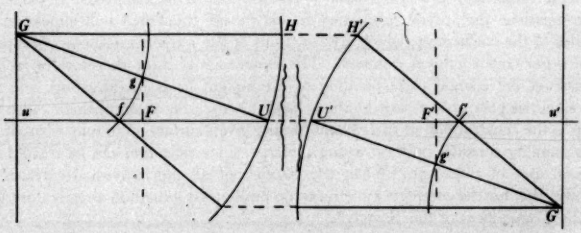

Fig. 29. The construction of conjugate secondary surfaces for a corrected lens

centres. The character of the imagery is illustrated by Fig. 29, which is drawn for negative unit magnification. The unit surfaces for rays which pass through G and G' are the spheres through U and U'. The principal focal surfaces are concentric spheres of half the radius. The figure shows that the ray through U and U' meets the focal spheres in points lying in the normal planes through F and F', the paraxial principal foci. The paths of rays through the system may be constructed exactly as in the conventional drawing shown in Fig. 2, the planes being replaced by spheres.

* *Trans. Opt. Soc.* **23** 1921–2 (311).

If the semi-aperture of the lens is equal to Gu, the longitudinal spherical aberration for a beam incident parallel to the axis is equal to fF or $F'f'$. The general image confusion outside the corrected planes is further illustrated in Fig. 30, which is the physically possible counterpart of the conventional imagery of Fig. 1.

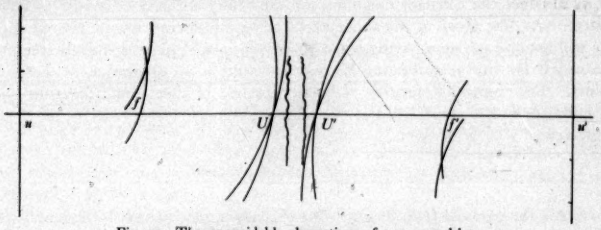

Fig. 30. The unavoidable aberrations of a corrected lens

Construction of the Eikonal

The foregoing discussion shows how the eikonal may be employed to derive all the properties of the system to which it relates by uniform processes. For general researches it is only necessary to assume that the eikonal exists. In particular investigations the special properties of the system concerned will appear in the values of the coefficients, which are functions of the refractive indices, curvatures, and other constructional elements. The lens designer must therefore be able to construct the eikonal corresponding to the system he is investigating, and it is an essential part of our task to show how this may be accomplished. The first step is the construction of the eikonal for any given surface. We may without loss of generality consider only refracting surfaces, since reflection can be treated as a special case of refraction. When the eikonals of all the individual surfaces are known, that for the complete system can be built up by a method already described in the *Transactions* of this Society*.

The eikonal for a single refracting surface

Suppose that the cartesian equation of the surface is $f(x, y, z) = 0$, and that the velocities of light on the two sides of the surface are v and v' respectively. In general this equation will not be homogeneous in form. Let it be expressed in the alternative form

$$f(a, x, y, z) = 0 \qquad \ldots\ldots(1),$$

where the equation is now homogeneous in a, x, y, z; if necessary this will be secured by the introduction of a to represent the unit of length. Let the reference planes for the two media both pass through the origin of coordinates and have

* *Trans. Opt. Soc.* **25** 1924 (177)

direction cosines (L, M, N) and (L', M', N') respectively. Since for the stationary path the light travels perpendicularly to these planes, the distances of (x, y, z) from these planes for the light are $Lx + My + Nz$ and $L'x + M'y + N'z$ respectively. It readily follows that, with the ordinary sign conventions, the time τ taken by the light to travel from the plane (L, M, N) to the plane (L', M', N') via the point (x, y, z) is given by

$$\tau = \frac{Lx + My + Nz}{v} - \frac{L'x + M'y + N'z}{v'} \qquad \text{.....(2).}$$

If (x, y, z) determines the neighbourhood of the surface for which the time is stationary between the planes, we must have

$$\left(\frac{L}{v} - \frac{L'}{v'}\right)\delta x + \left(\frac{M}{v} - \frac{M'}{v'}\right)\delta y + \left(\frac{N}{v} - \frac{N'}{v'}\right)\delta z = 0$$

for all infinitesimal values of δx, δy, δz which satisfy

$$f_x \delta x + f_y \delta y + f_z \delta z = 0.$$

That is to say we shall have

$$\frac{\frac{L}{v} - \frac{L'}{v'}}{f_x} = \frac{\frac{M}{v} - \frac{M'}{v'}}{f_y} = \frac{\frac{N}{v} - \frac{N'}{v'}}{f_z},$$

and each of these will be equal to

$$\frac{\frac{Lx + My + Nz}{v} - \frac{L'x + M'y + N'z}{v'}}{xf_x + yf_y + zf_z}.$$

Now, since f is homogeneous in a, x, y, z,

$$af_a + xf_x + yf_y + zf_z = 0,$$

and therefore by (2)

$$\frac{\tau}{af_a} = \frac{\frac{L'}{v'} - \frac{L}{v}}{f_x} = \frac{\frac{M'}{v'} - \frac{M}{v}}{f_y} = \frac{\frac{N'}{v'} - \frac{N}{v}}{f_z} \qquad \text{.....(3).}$$

If we divide throughout by T, the time of a complete vibration, we obtain

$$\frac{n}{af_a} = \frac{\frac{L'}{\lambda'} - \frac{L}{\lambda}}{f_x} = \frac{\frac{M'}{\lambda'} - \frac{M}{\lambda}}{f_y} = \frac{\frac{N'}{\lambda'} - \frac{N}{\lambda}}{f_z},$$

where n is the number of vibrations which occur as the light travels between the two reference planes, and λ and λ' are the wave-lengths of the light in the two media. If we multiply these equations throughout by the wave-length of the light in a standard medium we find

$$\frac{\mathcal{E}}{af_a} = \frac{\mu'L' - \mu L}{f_x} = \frac{\mu'M' - \mu M}{f_y} = \frac{\mu'N' - \mu N}{f_z} \qquad \text{.....(4),}$$

where $\mathscr{E}$ is the length of the equivalent path in the standard medium, and, since the wave-length is inversely proportional to the refractive index, μ and μ' are the refractive indices of the two media relative to the standard medium.

Now f_a, f_x, f_y, f_z are four homogeneous functions of a, x, y, z, between which the three ratios of these variables may be eliminated, giving

$$\phi(f_a, f_x, f_y, f_z) = 0 \qquad \ldots\ldots(5),$$

where ϕ is a homogeneous function. It at once follows that $\mathscr{E}$ satisfies the equation

$$\phi(\mathscr{E}/a, \mathscr{L}, \mathscr{M}, \mathscr{N}) = 0 \qquad \ldots\ldots(6),$$

where $\mathscr{L}, \mathscr{M}, \mathscr{N}$ denote $\mu'L' - \mu L$, $\mu'M' - \mu M$, $\mu'N' - \mu N$ respectively. This equation expresses $\mathscr{E}$, the stationary path length between the planes (L, M, N) and (L', M', N'), in terms of their direction cosines, so that $\mathscr{E}$ is the eikonal. If f is given in a finite form, the equation ϕ can also be obtained in a finite form, but whether this is possible for $\mathscr{E}$ also depends on the nature of the roots of ϕ. In practical cases there is no difficulty in selecting the correct root of ϕ. Since ϕ is finite $\mathscr{E}$ can be obtained to any assigned degree of accuracy for given numerical values of $\mathscr{L}, \mathscr{M}, \mathscr{N}$.

The use of the theorem will be seen from a few examples.

1. Consider refraction at the plane

$$lx + my + nz + p = 0.$$

Here p is a length which makes the equation homogeneous. Evidently

$$\frac{f_x}{l} = \frac{f_y}{m} = \frac{f_z}{n} = \frac{f_p}{1},$$

and since the direction cosines satisfy the condition

$$l^2 + m^2 + n^2 - 1 = 0,$$

we have

$$lf_x + mf_y + nf_z - f_p = 0.$$

The eikonal is thus given by

$$\mathscr{E} = p(l\mathscr{L} + m\mathscr{M} + n\mathscr{N}).$$

As an immediate corollary we have the result that the introduction of a plane parallel plate of a material of index μ' into a medium of index μ increases the optical path of the light by $t(\mu' \cos\phi' - \mu \cos\phi)$, where t is the thickness of the plate and ϕ and ϕ' are the angles made by the surfaces of the plate with the wave-fronts in the two media respectively.

2. Refraction at the paraboloid of revolution

$$y^2 + z^2 - 4ax = 0.$$

Here

$$\frac{f_a}{-4x} = \frac{f_x}{-4a} = \frac{f_y}{2y} = \frac{f_z}{2z},$$

so that

$$f_y^2 + f_z^2 - f_a f_x = 0$$

and

$$\mathscr{E} = a\frac{\mathscr{M}^2 + \mathscr{N}^2}{\mathscr{L}}.$$

In dealing with surfaces of revolution we may note that, if we write $y^2 + z^2 = w^2$, $\mathscr{M}^2 + \mathscr{N}^2 = \mathscr{W}^2$, corresponding to the relation $\phi(f_a, f_x, f_w) = 0$ we obtain the equation satisfied by the eikonal in the form $\phi(\mathscr{E}/a, \mathscr{L}, \mathscr{W}) = 0$.

3. Refraction at the spherical surface

$$x^2 + y^2 + z^2 - a^2 = 0.$$

In this case
$$\frac{f_x}{x} = \frac{f_y}{y} = \frac{f_z}{z} = \frac{f_a}{-a},$$

and therefore
$$f_x^2 + f_y^2 + f_z^2 - f_a^2 = 0,$$

or
$$\mathscr{E} = \pm a(\mathscr{L}^2 + \mathscr{M}^2 + \mathscr{N}^2)^{\frac{1}{2}}.$$

One root corresponds to the part of the surface which is convex, the other to the part which is concave, to the incident light.

4. Refraction at the ellipsoid

$$ax^2 + by^2 + cz^2 - 1 = 0.$$

Introduce the dummy unit variable t by writing the equation as

$$ax^2 + by^2 + cz^2 - t^2 = 0,$$

so that
$$\frac{f_t}{-t} = \frac{f_x}{ax} = \frac{f_y}{by} = \frac{f_z}{cz}$$

and
$$f_t^2 = \frac{f_x^2}{a} + \frac{f_y^2}{b} + \frac{f_z^2}{c}.$$

The value of the eikonal is thus

$$\mathscr{E} = \pm\left(\frac{\mathscr{L}^2}{a} + \frac{\mathscr{M}^2}{b} + \frac{\mathscr{N}^2}{c}\right)^{\frac{1}{2}}.$$

5. It is not essential for the surfaces to have finite curvature everywhere. Consider the semi-cubical paraboloid

$$27aw^2 - 4x^3 = 0$$

illustrated in Fig. 31, which has a singularity on the axis. Then

$$f_a = 27w^2, \quad f_w = 54aw, \quad f_x = -12x^2.$$

On substituting in the relation $(27aw^2)^2 = (4x^3)^2$ derived from the equation we obtain

$$f_a f_w^2 + f_x^3 = 0.$$

The expression for the eikonal is thus

$$\mathscr{E} = -a\frac{\mathscr{L}^3}{\mathscr{M}^2 + \mathscr{N}^2}.$$

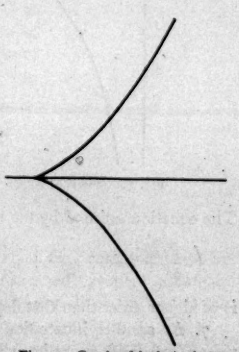

Fig. 31. Semi-cubical parabola

6. The following example shows that the equations f and ϕ satisfied by the surface and the eikonal respectively are not necessarily of the same order, and illustrates the formation of ϕ from

two non-homogeneous equations involving the differential coefficients of f and constants.

The surface

$$x(w^2 + a^2) - a^3 = 0$$

formed by the rotation of the Witch of Agnesi, shown in Fig. 32, about its axis, resembles a plane with a bulge near the origin: axial sections have a point of inflexion at $x = 3a/4$. Then

$$f_a = 2ax - 3a^2, \quad f_w = 2xw, \quad f_x = (w^2 + a^2),$$

so that

$$f_a f_x + 3a^2 f_x - 2a^4 = 0$$

and

$$f_a^2 + f_w^2 + 4a^2 f_a + 3a^4 = 0.$$

Eliminating a^2 gives

$$\{2f_a^2 + 2f_w^2 + 3f_a f_x\}^2 - f_x(8f_a + 9f_x)(f_a^2 - 3f_w^2) = 0.$$

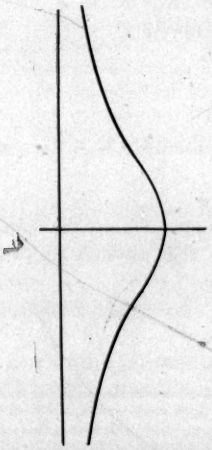

Fig. 32. Witch of Agnesi

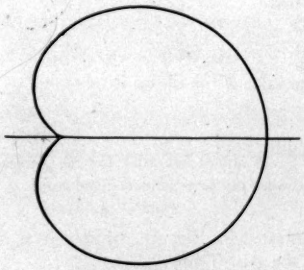

Fig. 33. Cardioid

The equation satisfied by the eikonal, viz.

$$\left(2\frac{\mathscr{E}^2}{a^2} + 2\mathscr{M}^2 + 2\mathscr{N}^2 + 3\frac{\mathscr{E}}{a}\mathscr{L}\right)^2 - \mathscr{L}\left(8\frac{\mathscr{E}}{a} + 9\mathscr{L}\right)\left(\frac{\mathscr{E}^2}{a^2} - 3\mathscr{M}^2 - 3\mathscr{N}^2\right) = 0$$

is of higher order than that defining the refracting surface.

7. As another illustration take the surface formed by rotating the cardioid $r = 2a(1 + \cos\theta)$ about its axis (see Fig. 33). The equation to the surface may be

written $r^2 - 2a(x + r) = 0$, where $r^2 = x^2 + y^2 + z^2$. When expressed in normal cartesian form the equation is of the fourth order. We have

$$f_a = -2(x + r),$$

$$f_x = 2x\left(1 - \frac{a}{r}\right) - 2a = 2x - r,$$

$$f_y = 2y\left(1 - \frac{a}{r}\right),$$

$$f_z = 2z\left(1 - \frac{a}{r}\right),$$

so that

$$f_a + f_x + 3r = 0,$$

$$af_a + r^2 = 0,$$

and

$$(f_x + 2a)^2 + f_y^2 + f_z^2 = 4(r - a)^2,$$

leading readily to

$$27 f_a (f_x^2 + f_y^2 + f_z^2) = 4(f_a + f_x)^3.$$

The equation for the eikonal

$$27 \frac{\mathscr{E}}{a}(\mathscr{L}^2 + \mathscr{M}^2 + \mathscr{N}^2) = 4\left(\frac{\mathscr{E}}{a} + \mathscr{L}\right)^3$$

is in this case of lower order than the cartesian equation to the surface.

8. In general the equation satisfied by the eikonal is most readily expressed as a determinant. Consider for example the general surface of the second order

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ix + 2jy + 2kz + d = 0.$$

On introducing the dummy variable t we obtain the five equations

$$\begin{aligned}
ax + hy + gz + it - \tfrac{1}{2}f_x &= 0,\\
hx + by + fz + jt - \tfrac{1}{2}f_y &= 0,\\
gx + fy + cz + kt - \tfrac{1}{2}f_z &= 0,\\
ix + jy + kz + dt - \tfrac{1}{2}f_t &= 0,\\
f_x x + f_y y + f_z z + f_t t &= 0,
\end{aligned}$$

from which it follows that the equation for the eikonal is

$$\begin{vmatrix}
a & h & g & i & \mathscr{L}\\
h & b & f & j & \mathscr{M}\\
g & f & c & k & \mathscr{N}\\
i & j & k & d & \mathscr{E}\\
\mathscr{L} & \mathscr{M} & \mathscr{N} & \mathscr{E} & .
\end{vmatrix} = 0.$$

9. For general surfaces of higher orders the order of the determinant will be much higher. This can be seen by considering a restricted group of equations of the third order, the limitation being that the equation is linear in one of the four variables a, x, y, z. We will refer to this particular variable as t. The total number of terms in f is then sixteen, and we may form a determinant of the

twentieth order by taking as independent quantities for elimination the sixteen third order products in f together with a, x, y, z. The twenty conditions are the equation to the surface in the form $af_a + xf_x + yf_y + zf_z = 0$; the six equations found by multiplying this equation throughout by all the second order combinations of a, x, y, z which do not involve t; and the thirteen equations giving the values of the sixteen products $uf_v\,(u, v = a, x, y, z)$ excluding the three $tf_v\,(v \neq t)$. The resulting identity is of the ninth order in f_a, f_x, f_y, f_z but not of more than the eighth order in any one of them. The equation for $\mathscr{E}$ is thus at most of the eighth order.

It has been mentioned that ϕ is a homogeneous function of f_a, f_x, f_y, f_z, but when f is of the first order, and its derivatives consequently of zero order, this result is not obvious. The conclusion may be inferred in this case by multiplying the original equation throughout by an arbitrary power of a, and is also clear from the following considerations, which indicate a procedure which can often be employed with advantage.

We are given five equations—the equation to the surface, and the expressions for its four derivatives. From these two independent equations connecting the four derivatives can be found by eliminating the three ratios $x : y : z : a$. These equations, say

$$\psi_1(f_a, f_x, f_y, f_z) = 0$$

and

$$\psi_2(f_a, f_x, f_y, f_z) = 0,$$

will not in general be homogeneous. If either is homogeneous the problem is solved. If neither is homogeneous they indicate that the two equations

$$\psi_1(tf_a, tf_x, tf_y, tf_z) = 0$$

and

$$\psi_2(tf_a, tf_x, tf_y, tf_z) = 0$$

have a common root, viz. $t = 1$. The condition that a common root exists—found by eliminating t between these two equations—is then satisfied, and this eliminant is obviously homogeneous in f_a, f_x, f_y, f_z. It is therefore the equation required.

10. This procedure may be illustrated by finding the eikonal for a surface of considerable importance in optics, the cartesian ovaloid

$$\alpha r + \beta r' - a = 0,$$

where

$$(x - ap)^2 + y^2 + z^2 = r^2,$$

$$(x - aq)^2 + y^2 + z^2 = r'^2.$$

Subtracting these equations we obtain

$$\begin{aligned} a(p - q)(2x - ap - aq) &= r'^2 - r^2 \\ &= \left(\frac{r'}{\beta} - \frac{r}{\alpha}\right)(\alpha r + \beta r') - \frac{(\alpha^2 - \beta^2)\, rr'}{\alpha\beta} \\ &= a\left(\frac{r'}{\beta} - \frac{r}{\alpha}\right) - (\alpha^2 - \beta^2)\frac{rr'}{\alpha\beta}. \end{aligned}$$

This equation suggests that we should alter the notation by writing

$$\frac{r}{a\alpha} = s, \quad \frac{r'}{a\beta} = s',$$

when it becomes

$$c\left(\frac{2x}{a} - p - q\right) = s' - s - (\alpha^2 - \beta^2)\, ss' \qquad \text{.....(1)},$$

where $c = p - q$ is the ratio of the distance between the two foci to the length a. The equation to the surface in this notation is

$$\alpha^2 s + \beta^2 s' - 1 = 0 \qquad \text{.....(2)}.$$

The differential coefficients are given by

$$-af_a = p\frac{x - ap}{s} + q\frac{x - aq}{s'} + a,$$

$$af_w = w\left(\frac{1}{s} + \frac{1}{s'}\right),$$

$$af_x = \frac{x - ap}{s} + \frac{x - aq}{s'}.$$

Write

$$\left.\begin{aligned} P &= pf_x + f_a, \quad Q = qf_x + f_a \\ R^2 &= f_w^2 + f_x^2 = f_x^2 + f_y^2 + f_z^2 \end{aligned}\right\} \qquad \text{.....(3)}.$$

Then

$$\left.\begin{aligned} P + 1 &= \frac{c}{s'} \cdot \frac{x - aq}{a} \\ Q + 1 &= -\frac{c}{s} \cdot \frac{x - ap}{a} \end{aligned}\right\} \qquad \text{.....(4)},$$

and eliminating x

$$(Q + 1)\, s + (P + 1)\, s' - c^2 = 0 \qquad \text{.....(5)}.$$

Again substituting for x from (4) in (1) gives

$$\frac{P}{s} - \frac{Q}{s'} + \alpha^2 - \beta^2 = 0 \qquad \text{.....(6)},$$

and, on substituting for w and x, the equation $af_a + wf_w + xf_x = 0$ becomes

$$\frac{P}{s} + \frac{Q}{s'} + R^2 = 0 \qquad \text{.....(7)}.$$

Of the four equations found connecting s and s' it will be noted that only the equation to the surface (2) is homogeneous. Disregarding homogeneity, equations (2) and (5) give

$$\frac{s}{P + 1 - \beta^2 c^2} = -\frac{s'}{Q + 1 - \alpha^2 c^2} = \frac{1}{(P + 1)\,\alpha^2 - (Q + 1)\,\beta^2},$$

and equations (6) and (7)

$$\frac{2P}{s} = -(R^2 + \alpha^2 - \beta^2),$$

$$\frac{2Q}{s'} = -(R^2 - \alpha^2 + \beta^2),$$

so that the two relations

$$\frac{2P}{P+1-\beta^2c^2}+\frac{R^2+\alpha^2-\beta^2}{(P+1)\alpha^2-(Q+1)\beta^2}=0,$$

$$\frac{2Q}{Q+1-\alpha^2c^2}-\frac{R^2-\alpha^2+\beta^2}{(P+1)\alpha^2-(Q+1)\beta^2}=0$$

are satisfied. When fractions are removed these are both of the third order in P, Q, R. Now if $|\,hk\,|$ denote the determinant

$$\begin{vmatrix} a_h & a_k \\ b_h & b_k \end{vmatrix}$$

the condition that the equations

$$a_1+a_2t+a_3t^2+a_4t^3=0,$$
$$b_1+b_2t+b_3t^2+b_4t^3=0$$

have a common root is

$$\begin{vmatrix} |12| & |13| & |14| \\ |13| & |14|+|23| & |24| \\ |14| & |24| & |34| \end{vmatrix}=0.$$

In our particular case $|23|=3|14|$, and, if

$$p'=1-\alpha^2c^2, \quad q'=1-\beta^2c^2, \quad S^2=PQ,$$
$$U=Pp'-Qq', \quad V=P\alpha^2-Q\beta^2, \quad W=Pp'+Qq',$$
$$\xi=p'q', \quad \eta=p'\beta^2+q'\alpha^2, \quad \zeta=\alpha^2-\beta^2,$$

we have

$$|12|=3\zeta^2U,$$
$$|13|=2\zeta(UV+\xi R^2),$$
$$|14|=\zeta WR^2,$$
$$|24|=6\zeta S^2R^2,$$
$$|34|=4VS^2R^2-UR^4,$$

so that the equation may be written

$$\begin{vmatrix} 3U & UV+\xi R^2 & WR^2 \\ UV+\xi R^2 & \zeta WR^2 & 3\zeta S^2R^2 \\ W & 3\zeta S^2 & 4VS^2-UR^2 \end{vmatrix}=0.$$

Now $$\zeta W=2\xi V-\eta U, \quad W^2=U^2+4\xi S^2,$$

and so the equation, on the removal of the factor U, which by the first example corresponds to refraction at a plane, becomes

$$4S^2UV^3-\{U^2V^2+28\xi S^2V^2-18\eta S^2UV-27\zeta^2S^4\}R^2$$
$$+2\{3\xi UV-2\eta U^2+\xi\eta S^2\}R^4-\xi^2R^6=0,$$

where the symbols now bear the following meanings:

$$c = p - q, \qquad R^2 = \mathscr{L}^2 + \mathscr{M}^2 + \mathscr{N}^2,$$
$$\xi = (1 - \alpha^2 c^2)(1 - \beta^2 c^2), \qquad V = \zeta \mathscr{E}/a + (p\alpha^2 - q\beta^2)\mathscr{L},$$
$$\eta = \alpha^2 + \beta^2 - 2\alpha^2\beta^2 c^2, \qquad U = c(\mathscr{L} - cV),$$
$$\zeta = \alpha^2 - \beta^2, \qquad S^2 = (V - \alpha^2 c\mathscr{L})(V - \beta^2 c\mathscr{L})/\zeta^2.$$

The equation for the eikonal is of the sixth, and that of the surface of the fourth order.

It will often be more convenient to obtain an explicit solution for the eikonal than to employ the finite equation it satisfies. The following form of the solution for surfaces of revolution is particularly useful. Let the equation to the surface, the axial radius of curvature of which is r, be given in the form

$$2x = r\phi(t), \quad w^2 = y^2 + z^2 = r^2 t.$$

Then
$$f_r = \phi - 2t\phi', \quad f_w = +2t^{\frac{1}{2}}\phi', \quad f_x = -2,$$

so that
$$\frac{f_r}{f_x} = t\phi' - \tfrac{1}{2}\phi, \quad \frac{f_w^2}{f_x^2} = t\phi'^2.$$

f_r/f_x may be expanded in a series of ascending powers of f_w^2/f_x^2 by Burmann's theorem. The coefficient of f_w^{2n}/f_x^{2n} in the expansion is

$$\frac{1}{n!}\frac{d^{n-1}}{dt^{n-1}}\left\{\left(\frac{t}{t\phi'^2}\right)^n \frac{d}{dt}\left(t\phi' - \tfrac{1}{2}\phi\right)\right\}_{t=0}$$
$$= \frac{1}{n!}\frac{d^{n-1}}{dt^{n-1}}\left\{\frac{1}{2\phi'^{2n-1}} + \frac{t\phi''}{\phi'^{2n}}\right\}_{t=0}$$
$$= \frac{1}{n!}\left\{\frac{1}{2}\frac{d^{n-1}}{dt^{n-1}}\frac{1}{\phi'^{2n-1}} + (n-1)\frac{d^{n-2}}{dt^{n-2}}\frac{\phi''}{\phi'^{2n}}\right\}_{t=0}$$
$$= \frac{1}{2(2n-1).n!}\frac{d^{n-1}}{dt^{n-1}}\left(\frac{1}{\phi'^{2n-1}}\right)_{t=0}$$
$$= \frac{1}{2n(2n-1)} \times \text{coefficient of } t^{n-1} \text{ in expansion of } 1/\phi'^{2n-1} \text{ in ascending powers of } t.$$

If then we put

$$\phi(t) = \frac{2x_0}{r} + t + \frac{a_2}{2}t^2 + \frac{a_3}{3}t^3 + \frac{a_4}{4}t^4 + \ldots,$$

we shall have for the eikonal

$$2\mathscr{E} = r\mathscr{L}\psi(T), \quad T = \frac{\mathscr{M}^2 + \mathscr{N}^2}{\mathscr{L}^2},$$

$$\psi(T) = -\frac{2x_0}{r} + T + \frac{b_2}{2}T^2 + \frac{b_3}{3}T^3 + \frac{b_4}{4}T^4 + \ldots,$$

where b_n is the coefficient of t^{n-1} in the expansion of

$$\frac{(1 + a_2 t + a_3 t^2 + a_4 t^3 + \ldots)^{1-2n}}{2n-1}$$

in ascending powers of t. It is thus a simple matter to determine the eikonal to any desired number of terms. The form of equation from which we started is always valid for the extent of surface in which we are interested for optical purposes, and the conversion of the equation to this form is easy.

The refracting surface for a given eikonal

It will have become clear from this discussion that the eikonal for a single surface is always a homogeneous function of order one in $\mathscr{L}, \mathscr{M}, \mathscr{N}$. We shall now show that the surface can be found which has an arbitrarily given function of this form for its eikonal.

From a fundamental property of the eikonal established earlier, the points

$$\left(0, \frac{\partial \mathscr{E}}{\partial (\mu M)}, \frac{\partial \mathscr{E}}{\partial (\mu N)}\right) \text{ and } \left(0, -\frac{\partial \mathscr{E}}{\partial (\mu' M')}, -\frac{\partial \mathscr{E}}{\partial (\mu' N')}\right)$$

lie on the stationary path between the planes (L, M, N) and (L', M', N'). If $\mathscr{E}$ is a function of $\mathscr{L}, \mathscr{M}, \mathscr{N}$ only, these points are

$$\left(0, -\mathscr{E}_{\mathscr{M}} + \frac{M}{L}\mathscr{E}_{\mathscr{L}}, -\mathscr{E}_{\mathscr{N}} + \frac{N}{L}\mathscr{E}_{\mathscr{L}}\right) \text{ and } \left(0, -\mathscr{E}_{\mathscr{M}} + \frac{M'}{L'}\mathscr{E}_{\mathscr{L}}, -\mathscr{E}_{\mathscr{N}} + \frac{N'}{L'}\mathscr{E}_{\mathscr{L}}\right).$$

Since the directions in which the light travels are (L, M, N) and (L', M', N') respectively, it is obvious that both paths pass through the point

$$(-\mathscr{E}_{\mathscr{L}}, -\mathscr{E}_{\mathscr{M}}, -\mathscr{E}_{\mathscr{N}}) \equiv (\xi, \eta, \zeta).$$

One of the essential conditions for the system to be equivalent to single surface refraction is therefore satisfied. Now $\mathscr{E}$ by hypothesis satisfies a homogeneous equation

$$\phi(\mathscr{E}/a, \mathscr{L}, \mathscr{M}, \mathscr{N}) = 0,$$

where $\mathscr{L}, \mathscr{M}, \mathscr{N}$ may be regarded as independent variables. Varying each in turn, and inserting the values of ξ, η, ζ, we obtain

$$\frac{\phi_{\mathscr{E}/a}}{a} = \frac{\phi_{\mathscr{L}}}{\xi} = \frac{\phi_{\mathscr{M}}}{\eta} = \frac{\phi_{\mathscr{N}}}{\zeta}.$$

Now $\phi_{\mathscr{E}/a}, \phi_{\mathscr{L}}, \phi_{\mathscr{M}}, \phi_{\mathscr{N}}$ are homogeneous functions of $\mathscr{E}/a, \mathscr{L}, \mathscr{M}, \mathscr{N}$, and by eliminating the three independent ratios we can form an equation

$$f(\phi_{\mathscr{E}/a}, \phi_{\mathscr{L}}, \phi_{\mathscr{M}}, \phi_{\mathscr{N}}) = 0.$$

It follows that ξ, η, ζ satisfy the equation

$$f(a, \xi, \eta, \zeta) = 0.$$

This is evidently the equation of a surface through the common points of the various paths. To show that we may suppose this to be a real refracting surface it is only necessary to observe that the homogeneous relation

$$\frac{\mathscr{E}}{a}\phi_{\mathscr{E}/a} + \mathscr{L}\phi_{\mathscr{L}} + \mathscr{M}\phi_{\mathscr{M}} + \mathscr{N}\phi_{\mathscr{N}} = 0$$

also involves

$$\mathscr{E} + \mathscr{L}\xi + \mathscr{M}\eta + \mathscr{N}\zeta = 0,$$

that is to say the value of the given path is exactly equal to that resulting from refraction at the surface $f(a, \xi, \eta, \zeta) = 0$. In other words if the eikonal is given we can find the equation of the corresponding surface.

This result is of great importance, for it indicates that by Hamiltonian methods we are not restricted to the customary procedure of finding the properties of given systems, but may impose the conditions to be observed and find a system which satisfies them. No attempt seems to have been made hitherto to utilise this method in the design of lenses.

The process by which the eikonal is formed from the equation of the surface, it will be observed, has its exact counterpart in the determination of the surface from the eikonal. It follows that if the surface $f(a, x, y, z) = 0$ has $\phi(\mathscr{E}|a, \mathscr{L}, \mathscr{M}, \mathscr{N}) = 0$ for its eikonal equation, then the surface $\phi(b, x, y, z) = 0$ has $f(\mathscr{E}|b, \mathscr{L}, \mathscr{M}, \mathscr{N}) = 0$ for its eikonal equation.

As an example of this reciprocal property we observe that, in the general expansion of the eikonal in series form given above, the relations between the a's and b's are symmetrical, so that a_n is the coefficient of T^{n-1} in the expansion of

$$\frac{(1 + b_2T + b_3T^2 + b_4T^3 + \ldots)^{1-2n}}{2n - 1}$$

in ascending powers of T.

ABERRATION COEFFICIENTS OF A SIMPLE LENS

Although the method of constructing the eikonal of a compound system from those of its parts has been described before, it will not be without interest to determine in this way the fundamental aberrational coefficients of a thick lens on the model of Airy's formulae for a thin lens. In addition to illustrating the readiness with which the method yields results of value, the expressions so found may be expected to suggest some of the conclusions already derived by geometrical considerations.

The eikonal for a compound instrument is obtained by adding together those of its parts. The intermediate variables are eliminated by giving them the values for which the new eikonal is stationary. For the addition of two symmetrical systems of powers κ_1 and κ_2 to make a combination of power κ, the values required are at once found as

$$\mu_1 M_1 \kappa = \mu_0 M_0 \kappa_2 + \mu_2 M_2 \kappa_1,$$
$$\mu_1 N_1 \kappa = \mu_0 N_0 \kappa_2 + \mu_2 N_2 \kappa_1.$$

By the Smith-Helmholtz law μMy is invariant on refraction, and therefore, if s is the paraxial magnification produced by the second surface corresponding to S for the complete lens, these formulae give

$$s\kappa = \kappa_1 + S\kappa_2,$$

and similarly for another magnification

$$g\kappa = \kappa_1 + G\kappa_2.$$

We shall take G as the magnification for the image of which the aberrations are to be found, and S that of the stop. To secure the greatest symmetry we adopt $S = 1$, $G = -1$ as the standard reference system. It is at once seen that s is a factor which will enter into the Airy-Petzval coefficient, and will also determine the thickness of the lens, while g becomes a variable giving the shape of the lens. In terms of these two variables

$$\kappa_1 = \tfrac{1}{2}(s+g)\,\kappa$$

and

$$\kappa_2 = \tfrac{1}{2}(s-g)\,\kappa.$$

Also, if $\mu_0 = \mu_2 = 1$ and μ is written for μ_1, the transformation equations are

$$\mu M_1 = \tfrac{1}{2}s\,(M_2 + M_0) + \tfrac{1}{2}g\,(M_2 - M_0),$$
$$\mu N_1 = \tfrac{1}{2}s\,(N_2 + N_0) + \tfrac{1}{2}g\,(N_2 - N_0),$$

so that if, in accordance with the standard means of defining aberrations, we write

$$(M_2 + M_0)^2 + (N_2 + N_0)^2 = 4A,$$
$$(M_2 + M_0)(M_2 - M_0) + (N_2 + N_0)(N_2 - N_0) = 4B,$$
$$(M_2 - M_0)^2 + (N_2 - N_0)^2 = 4C,$$

we shall have

$$\mu(M_0M_1 + N_0N_1) = sA - (s-g)B - gC,$$
$$\mu(M_1M_2 + N_1N_2) = sA + (s+g)B + gC,$$
$$L_0 = \{1 - A + 2B - C\}^{\frac{1}{2}},$$
$$\mu L_1 = \{\mu^2 - s^2A - 2sgB - g^2C\}^{\frac{1}{2}},$$
$$L_2 = \{1 - A - 2B - C\}^{\frac{1}{2}}.$$

Now the eikonal e for a single spherical surface of power κ, with the reference points on the axis chosen to correspond to the paraxial magnification g, is given by

$$e\kappa = (\mu' - \mu/g)\,\mu L + (\mu - \mu' g)\,\mu' L' + (\mu' - \mu)\{\mu'^2 + \mu^2 - 2\mu'\mu\,(LL' + MM' + NN')\}^{\frac{1}{2}}.$$

Noting that the object and stop magnifications of the first surface are $-1/g$ and $1/s$, and those of the second g and s, it follows that the eikonal for our lens may be written, neglecting the constant term,

$$\begin{aligned}\tfrac{1}{2}e\kappa\,(s^2 - g^2) = {} & g\,(s-g)\,L_0 - 2\mu^2 L_1 - g\,(s+g)\,L_2 \\ & - \mu\,(s-g)\,\{(1-L_0)(1-L_1) + M_0M_1 + N_0N_1\} \\ & - \mu\,(s+g)\,\{(1-L_1)(1-L_2) + M_1M_2 + N_1N_2\} \\ & - \frac{(s-g)}{2\,(1-\omega)^2}\,\{1 - (L_0L_1 + M_0M_1 + N_0N_1)\}^2 \\ & - \frac{(s+g)}{2\,(1-\omega)^2}\,\{1 - (L_1L_2 + M_1M_2 + N_1N_2)\}^2 \\ & - \ldots,\end{aligned}$$

where ω denotes the reciprocal of μ. Changing the variables

$$\begin{aligned}
\tfrac{1}{2}\mathscr{E}\kappa\,(s^2-g^2) &= -(s^2-g^2)\,A - \tfrac{1}{4}\,(s^2-g^2)\,(A+C)^2 \\
&\quad + \tfrac{1}{4}\,\{sA + 2gB + sC - (s^2A + 2sgB + g^2C)\,\omega\}^2 \\
&- \frac{(s-g)}{8\,(1-\omega)^2}\,\{(1-s\omega)^2\,A - 2\,(1-s\omega)\,(1+g\omega)\,B + (1+g\omega)^2\,C\}^2 \\
&- \frac{(s+g)}{8\,(1-\omega)^2}\,\{(1-s\omega)^2\,A + 2\,(1-s\omega)\,(1-g\omega)\,B + (1-g\omega)^2\,C\}^2 \\
&\quad - \ldots \\
&= -(s^2-g^2)\,A + \tfrac{1}{4}\,[s\,(1-s\omega)^2\,(s-p^2) - (s^2-g^2)]\,A^2 \\
&\quad + g\,(1-s\omega)^2\,(s-p^2)\,AB - \tfrac{1}{2}s\omega\,(s^2-g^2)\,AC \\
&\quad + \frac{1}{s}\,[g^2\,(1-s\omega)^2\,(s-p^2) - p^2\,(s^2-g^2)]\,(B^2 + \tfrac{1}{2}AC) \\
&\quad + \frac{g}{s\,(1-s\omega)}\,[(s-g^2\omega)\,(1-s\omega)^2\,(s-p^2) + p^2\omega\,(1+s\omega)\,(s^2-g^2)]\,BC \\
&\quad + \frac{1}{4s\,(1-s\omega)^2}\,[g^2\,(1-2s\omega+g^2\omega^2)\,(1-s\omega)^2\,(s-p^2) \\
&\qquad - p^2\,\{1+g^2\omega^2\,(1+2s\omega)\}\,(s^2-g^2)]\,C^2 - \ldots,
\end{aligned}$$

where

$$p = \frac{1-s\omega}{1-\omega}.$$

Now put

$$\begin{aligned}
q &= \frac{(1-s\omega)^2\,(s-p^2)}{(s^2-g^2)\,p^2} \\
&= \frac{(s-1)\,(1-s\omega^2)}{s^2-g^2} \\
&= \tfrac{1}{4}\omega\,(1-s\omega^2)\,t\kappa.
\end{aligned}$$

q is evidently always a small quantity, attaining a value near to 0·03 for lenses in which the thickness suffices to make the aperture equal to the focal length. The Airy-Petzval coefficient is $s\omega$, which we will denote by ϖ. One of the instructive forms into which the formula may be cast enables a direct comparison to be made with a thin lens of the same shape made of material having the refractive index $1/\varpi$. Since g/s is independent of the refractive index, the normal variable for expressing the shape of this lens is ρ, where

$$2\rho = \frac{g}{s}\cdot\frac{\varpi}{1-\varpi} = \frac{g\omega}{1-s\omega}.$$

Then

$$\begin{aligned}
&(\tfrac{1}{2}\mathscr{E}\kappa + A + \tfrac{1}{4}A^2 + \ldots)\,\frac{s}{p^2} \\
&\qquad = \frac{1}{4}\,s^2qA^2 + sgqAB - \frac{1}{2}\,\frac{s}{p^2}\,\varpi AC - i\,(B^2 + \tfrac{1}{4}AC) \\
&\qquad\quad + 2\rho\,(i + j\varpi)\,BC - \frac{1}{4}\left\{\frac{i}{(1-\varpi)^2} + 4\rho^2\,(i+2j\varpi)\right\}C^2 \\
&\qquad\quad - \ldots,
\end{aligned}$$

where

$$i = 1 - g^2q, \quad j = 1 + \mu^2q.$$

The terms within the bracket on the left are those which correspond to the absence of aberrations*. The factor s/p^2 suggests that, except for the Airy-Petzval coefficient, the fictitious thin lens with which the comparison is made should have its focal length p^2 times that which the actual lens would have if its thickness were zero—since p differs little from unity this distinction is usually unimportant. Then the only differences to be found between the real and the fictitious lenses are the very small stop aberration and distortion, the mutual ratio of which depends on the shape of the lens alone and not on the refractive index, and the slight departures of i and j from unity. All these aberrations for any thick lens, it will be seen, correspond very closely to those of a thin lens. This property is of great importance to the lens designer.

It should hardly be necessary to add that these six coefficients form a complete set, i.e. they give the first order aberrations of the lens for any object position and any stop position. The way in which they are combined for this purpose has already been described*.

In comparing these algebraic results with the geometrical conclusions reached earlier, it should be noted that the curvature of the harmonic sphere is

$$2\rho\kappa s = 2\rho\,(\kappa_1 + \kappa_2).$$

The formulae we have established thus show that the curvature of this sphere is the outstanding factor on which the aberrations of the lens depend. It may be of interest to give a table of these algebraic quantities for a "thin" lens resembling "a," and for "a," "b," "c" and "d" of Fig. 15.

Lens	"thin a"	"a"	"b"	"c"	"d"
s	1	$\frac{15}{16}$	$\frac{15}{8}$	$\frac{5}{4}$	$\frac{15}{4}$
g	3	$\frac{45}{16}$	$\frac{5}{8}$	$-\frac{5}{12}$	$\frac{45}{4}$
p	1	$\frac{9}{8}$	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{9}{2}$
q	0	$\frac{7}{1350}$	$\frac{7}{150}$	$\frac{2}{225}$	$\frac{11}{675}$
ϖ	$\frac{2}{3}$	$\frac{5}{8}$	$\frac{5}{4}$	$\frac{5}{6}$	$\frac{5}{2}$
i	1	$\frac{391}{2197}$	$\frac{377}{384}$	$\frac{74}{78}$	$-\frac{14}{16}$
j	1	$\frac{607}{600}$	$\frac{221}{200}$	$\frac{52}{50}$	$\frac{311}{300}$
2ρ	6	5	$-\frac{5}{3}$	$-\frac{5}{3}$	5

THE BALANCING OF ABERRATIONS

The next field to which I would direct attention has only been opened out in recent years. The need for a careful balance of one aberration against another, as an important problem, arose with the evolution of wide aperture lenses covering a large angular field. The ultimate solution requires much careful visual observation and other tests; at present no more than a beginning has been made with this part of the problem. It is by no means certain that, at the present stage of development of optical theory, the school which advocates the manufacture of trial instruments when a design has reached an advanced stage are not building on a much

* *Trans. Opt. Soc.* **23** 1921–2 (311).

more secure foundation than those who hold that a design should be completed on paper and be immediately followed by mass production. The latter procedure is based on a number of assumptions which have been in the past regarded as established facts. In fact we know to-day that some are definitely incorrect. There can however be no doubt that the construction of experimental models is not a satisfactory ultimate solution of the problem. When suitable sets of observations on a systematic basis have been carried out the designer will know exactly by what criterion the quality of optical images is to be judged, and the attainment of the best result that is possible can then be secured by calculation alone.

At the present time we know that sensibly perfect images are secured when the disturbances reaching the image point from all parts of the lens aperture do not vary in phase by more than one quarter of a period. Such a standard is only reached by modern instruments over small fields. Over a part of the field indeed a distinctly higher standard may be attained, and this suggests that the best possible compromise for a larger field has not been obtained. Supposing that a length standard of this kind must be imposed, and that the instrument cannot be strictly corrected for all defects, how are the optimum amounts of the various aberrations to be ascertained?

For a single kind of aberration—say central spherical aberration—the solution is not difficult to find. The variation in the path will be a minimum when it varies between fixed limits as often as possible, reaching these limits for the extreme rays—in this case the axial and the marginal rays—and for each stationary zone. It is not difficult to show that any other kind of compromise involves a greater total range of phase. We may express this result by saying that the spherical aberration is represented by a function of minimum variation in the single variable involved. The general problem with which we are now concerned is much more difficult because the function to be dealt with involves more than one independent variable—there are in fact three in a symmetrical instrument. The solution of the problem on the old geometrical lines, where the various aberrations are treated separately, seems to be a sheer impossibility, and this quite apart from the failure of this method to give directly an acceptable measure of the fault to be balanced. It appears essential for success that regard should be paid simultaneously to all the defects present, so that by making a known function of several variables a function of minimum variation for each of them separately as well as all together, the best balance may be attained for the whole field and the whole aperture in a single operation.

It is obvious that there is in this problem nothing of an optical character, however important the optical application of a solution may be. It is one of those problems in pure mathematics by the solution of which the gratitude of the optician would be added to any other recompense the mathematician might gain. The solution appears to consist of the product of functions each of which is a function of minimum variation of a single variable, or perhaps we should rather say the sum of a number of such products. But there is one complication present in the actual optical problem which has not been taken into account. The solution

depends essentially on the limits reached by the variables, and these limits are not independent; for instance we are all aware that the aperture of any real lens is greatly reduced as we pass from the centre of the field towards the margins. The present indications are that full advantage must be taken of the smaller range of the aperture variables in such positions if the best compromise is to be reached.

This is neither the time nor the place in which to attempt any detailed discussion of the problem, which indeed only comes within the scope of this address owing to its optical urgency, but it may prove of interest to show graphically the distribution of the maxima and minima which are found in a function of two variables when the limits between which the variables lie are independent. Suppose we have to form a function of minimum variation with $x^m y^n$ as an unalterable term, where m and n are integers, and the variables x and y each range from -1 to 1. By adding other terms of the form $x^p y^q$ $(p < m, q < n)$ we can reduce the amplitude of the variation to 2^{2-m-n} of its original value. For we only have to put $x = \cos\theta$, $y = \cos\phi$, and take the function

$$2^{2-m-n} \cos m\theta \cos n\phi,$$

which is an integral function of $\cos\theta$ and $\cos\phi$ having $\cos {}^m\theta \cos {}^n\phi$ as its highest term, to make this conclusion evident. In passing it may be noted that the correction yielded by compensation on these lines would be quite unlike that attained in present day instruments. From the theory it is obvious that the variations attain their maximum range at the limit of either variable, and these points include the centre of the field. Optical instruments as we know them are always better corrected on the axis than elsewhere. Taking $m = 3$, $n = 5$ we obtain

$$x^3y^5 - 1{\cdot}25x^2y^3 - {\cdot}75xy^5 + {\cdot}3125x^3y + {\cdot}9375xy^3 - {\cdot}234375xy;$$

the distribution of points of maximum variation, the zero lines, and the contours for half the extreme values, are indicated in Fig. 34. The compensated range is only 1·6 per cent. of that reached by the uncompensated function at the margins. This numerical example illustrates the great effectiveness for the correction of lenses of power to control at will, and independently of one another, the aberrations of the lowest orders.

THE THEORY OF NARROW LENSES

This consideration brings me to the last of the fields about which I am going to inflict a few remarks upon you; the field of greatest interest to many of us, albeit the one on which I propose to say the least, the general procedure to be adopted in evolving better corrected lenses. If we may imagine the time has come when the various fields already mentioned have been well cultivated we shall start with a general knowledge, of fair quantitative accuracy, of the aberrations of lenses over the whole of their fields; we shall be armed with a most powerful method of investigating these aberrations with precision; we shall know what is under control, and what faults cannot be removed; we shall know how to make the best use of the degrees of freedom we possess, expressing this in advance by

assigning definite values to be attained by the aberrations under control: further we shall be able to find how the task of securing these values is to be divided among the elementary lenses of the system, and thence find the forms their surfaces should receive, and ultimately determine the glasses from which they are to be made. Such a procedure, having hardly a single feature in common with that followed now, may seem involved, but conceivably it may assume a remarkably simple form. Our aim, we may presume, is to design a lens with the largest possible aperture and largest field consistent with the attainment of a definite standard of definition everywhere, and we wish to reach the final form with as little trouble as

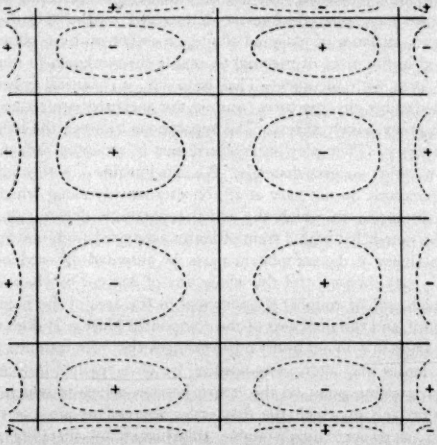

Fig. 34. Contours $\pm \frac{1}{64}$ (peak points marked + and −), $\pm \frac{1}{128}$ (dotted lines), and o (full lines) in compensated value of x^2y^6 $(-1 \leqslant \frac{x}{y} \leqslant 1)$

we may. To see how we are to proceed we will first travel in the reverse direction from this highest form to lower levels. We imagine then a lens of the largest admissible aperture, covering a wide field, and satisfying a criterion for freedom from aberrations such as we have discussed, to have been designed. Such a lens will not fall below the given standard of correction for smaller apertures or over less extensive fields. Under these less exacting conditions other forms equally good will be possible, the variations covering a wider and wider range as the aperture and field are reduced. We may therefore follow many tracks down from the summit through a continuous series of forms, reaching at the base lenses of such small apertures that aberrations of all but the lowest orders become insignificant. In general the large aperture lens will have appreciable, though not large, amounts of these low order aberrations present, but we may choose a path, such as one

along which the aberrations at all stages are represented by functions of minimum variation, to a terminal lens having its lowest order aberrations exactly corrected. But the region at this low level from which we may choose a starting point for the ascent is still embarassingly large. It happens that we may impose a further restriction on the downward path which limits the choice of a starting point on the return journey; this restriction moreover has the advantage of simplifying the calculations in the initial and subsequent stages alike; in addition it confines our attention to forms which it is advantageous to manufacture. The restriction rests on the assumption that in general the thickness of the glass in an optical instrument of the kind we are considering will not be utilised as a correcting element. The individual lenses have their thicknesses determined by the aperture that has to be attained with surfaces of assigned shapes, together with an allowance for the rigidity needed in the piece of material to enable the workman to impress upon it the required form with accuracy. That is to say, in practical constructions the thicknesses determine the apertures; and as the apertures are reduced the thicknesses should progressively shrink. The separations between the lenses belong to a different category. They play an essential part in attaining that distribution of aberrations on which success depends. We can imagine a path of descent along which the separations do not vary at all, or alternatively along which some more fundamental distances, on which the actual separations depend, do not change. Now it is to be noted that with a train of lenses separated from one another, as we reduce the apertures of the component parts we automatically reduce the field of view also. We may then control the whole line of descent by changes in a single variable t, which will be roughly proportional to the area of the field, the area of the lens aperture, and the thickness of the component lenses. It does not of course follow that changes in t do not imply other changes also—for instance the refractive indices of the lenses may alter. The position we have reached however is that we can take as our starting point on the return journey a lens in which all the components are thin and in which the first order aberrations are exactly corrected. The next step is to introduce suitable thicknesses by changing t from 0 to t_1. This causes second order aberrations to become sensible—let us represent them by $a^6\phi_2$ where a is the aperture of a zone of the lens. This change also alters the first order aberrations—instead of vanishing there are residuals proportional to t_1. The aberrations present are therefore of the form $a^4t_1\phi_1 + a^6\phi_2$. Now a varies from 0 to A_1, say, and t is proportional to A_1^2. The expression $a^6\phi_2 + a^4A_1^2\phi_1' + a^2A_1^4\phi_0$, where ϕ_0 represents adjustment of focus, can be made a function of minimum variation by securing values of ϕ_0 and ϕ_1' properly related to ϕ_2—a problem which is known to be soluble. We have now obtained a solution with the aperture A_1 *by using the thickness variation of the lower order aberrations to produce minimum variations of phase in the presence of higher order aberrations.* If the aberrations of the second order are under control we may continue the process farther, increasing the aperture to A_2 and taking into account the thickness effects of the lower order aberrations. When we reach a stage at which we cannot effectively control the new aberrations, we merely increase the aperture, and in so doing of

course increase the field of view also, until any further increase would involve a violation of the limiting condition which determines good correction.

There is one point of importance in this process to which attention should be directed. When a new aberration is introduced its actual value need not be considered, but only the value it would have if the component lenses were indefinitely thin. The effects of thickness need only be taken into account when aberrations of still higher orders are to be compensated. In itself this brings about a remarkable simplification in the calculations. But we may go farther, and carry out the whole process by calculations based on thin lenses. We have already seen that the first order aberrations of a thick lens are almost exactly representable by those of a thin lens of somewhat different refractive index and approximately the same focal length. Such analogies may be extended to aberrations of higher orders. They may enable us to calculate out the corrections for an entirely fictitious system of thin lenses, which, if constructed, would not yield an aperture or field of view of useful size. From this theoretical result we may be able to put down data for the construction of real instruments, having suitable thicknesses for the component lenses, and possibly differing in every other detail from the system used in the calculations, yet securing the same standard of performance as these calculations indicate.

It is customary to restrict the term "thin" to lenses whose thicknesses are entirely negligible. On the other hand by "thick" lenses we denote those in which the thicknesses may reach very great values. The lenses of real importance and interest are, as has been said before, those in which the thicknesses are only made great enough to yield the required apertures and to leave margins for mechanical strength. A word is needed to denote briefly lenses of this kind, and the best suggestion I can offer is to call them "narrow" lenses. If any should prefer they may call them "slim" lenses. In the foregoing paragraphs I have indicated that a special algebraic method of finding valuable combinations of lenses, with a special theory, can be constructed on the assumption that the elementary lenses used in an optical instrument are narrow lenses. This theory, of which only the barest outline has been indicated above, I propose to call the theory of narrow lenses. In this theory all the problems to which I have ventured to draw your attention have their place. Much remains to be done before it can be considered a suitable tool for general use; it is possible that it may fail to earn the approval of most computers. But at least it offers greater promise than attempts made in the past to design lenses entirely by algebraic processes, and the fact that in the only class of system in which it has been thoroughly tried out, the class consisting of a group of lenses in contact, the simplest conceivable procedure on these lines—employing in the real lens the curvatures found by solving the conditions for the first order aberrations with thin lenses*—is so successful, encourages the hope that some correspondingly simple procedure will be found applicable to other classes of instrument. Whether we think that success on such lines is to be achieved easily, or not at all; even if we believe that nothing is destined to replace or alter the present

* *Proc. Phys. Soc.* **30** 1918 (119).

method of design by the aid of ray tracings; it is nevertheless our business to see that none of the ground over which we have control remains unused. If we fail to utilise it, the onus lies on us of showing that we have attempted, with such resources as we possess, to bring it under cultivation, and have only abandoned such efforts when no doubt has been left in our minds that the land is infertile. My endeavour this evening has been, by exhibiting a few of the native fruits I have chanced to find, to encourage in you the hope that much of the ground now lying fallow will yield a rich return to those who till it.

[On reading the proof I notice that I have employed the axial displacement definition of focal length in the geometrical part of this address, and the magnification definition in the subsequent analytical sections. The two do not agree when the initial and final media differ, but the connexion between them is too well known for it to be necessary to do more than direct attention to this slight variation in the use of the term. T. S.]

THE STATIONARY VALUE OF AXIALLY SYMMETRIC FUNCTIONS: PART II

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MS. received, 9th October, 1926. Read, 28th October, 1926.

ABSTRACT—Alternative methods to those described in Part I of constructing a series representing the stationary value of a given function are developed and applied to evaluate all the terms not involving powers and products of the coefficients of the function higher than the eleventh. The formula in its optical applications enables the first 451 monorythmic aberrations of a symmetrical optical instrument to be determined.

INTRODUCTION

IN the first part* of this investigation it has been shown that if f is a function of the four variables x_1, x_2, x_3, x_4 and possesses symmetry of the type which enables us to regard f as a function of the three variables a, b, c, where

$$a = \tfrac{1}{2}(x_1^2 + x_2^2), \qquad b = x_1x_3 + x_2x_4, \qquad c = \tfrac{1}{2}(x_3^2 + x_4^2),$$

the value of f which satisfies the conditions

$$\frac{\partial f}{\partial x_h} = 0 \quad (h = 1, 2, 3, 4)$$

is expressible by a series of terms obtained by carrying out operations on v, the part of f which contains no terms of the first or second orders, denoted by the letters D and E. The definitions of these operations are

$$Dp = \{vp\}, \qquad Ep = \{up\}, \qquad u = Dv,$$

where

$$\{pq\} = \frac{1}{2(\beta^2 - \alpha\gamma)}\{(ap_a + bp_b + cp_c)(\gamma q_a - 2\beta q_b + \alpha q_c) + (\gamma p_a - 2\beta p_b + \alpha p_c)(aq_a + bq_b + cq_c) + (a\alpha + \beta b + \gamma c)(2p_bq_b - p_aq_c - p_cq_a)\},$$

and α, β, γ are the coefficients of a, b, c in f, and a, b, c are given special values. The method of expansion described in the first part has been used to obtain the stationary value up to the terms which involve the seventh powers of the coefficients of f. In this part other methods are described and applied to carry the expansion to a higher order.

* "The Stationary Value of Axially Symmetric Functions," Part I, *Trans. Opt. Soc.* **27** 1925–6 (307).

ALTERNATIVE METHOD

A somewhat different method from that described in Part I of carrying out the transformation consists in putting $q = v$ initially in all the equations except (6) and (7), which are used to perform the operations D and E on any of the quantities formed previously. At the same time the determination of new numerical coefficients at the substitution stage may be avoided. To effect this an alteration in the notation hitherto employed is helpful. The bracketed quantity $(p, q, r, s, t \ldots)$ will be used to denote a product term of f_s in the original expression* which contains p undifferentiated terms, q terms differentiated once, r terms differentiated twice, s differentiated three times, and so on. In place of (1) we then have the expression

$$\begin{aligned} f_s = f &+ (2) + (21) + (301) + (22) + (4001) + (311) + (23) \\ &+ (50001) + (4101) + (402) + (321) + (24) \\ &+ (600001) + (51001) + (5011) + (4201) + (412) + (331) + (25) \\ &+ \ldots \qquad \ldots\ldots(22), \end{aligned}$$

the coefficient of every term being unity by definition, so that the transformation from the old notation to the new has involved the absorption into the typical term of the numerical factor

$$\frac{(q + 2r + 3s + 4t \ldots)!}{p!\,(0!)^p\, q!\,(1!)^q\, r!\,(2!)^r\, s!\,(3!)^s\, t!\,(4!)^t \ldots} \qquad \ldots\ldots(23).$$

The quantities $p, q, r, s, t \ldots$ are not all independent, for they satisfy the relation

$$2 = -p + r + 2s + 3t + \ldots \qquad \ldots\ldots(24).$$

The application of operations (6) and (7) then gives

$$\begin{aligned} &\frac{2(1 + q + 2r + 3s + \ldots)}{1 + p} D(p, q, r, s, t \ldots) \\ &\quad = \frac{p(q+1)}{p+1}(p, q+1, r, s, t \ldots) \\ &\qquad + 2(r+1)(p+1, q-1, r+1, s, t \ldots) \\ &\qquad + 3(s+1)(p+1, q, r-1, s+1, t \ldots) \\ &\qquad + 4(t+1)(p+1, q, r, s-1, t+1 \ldots) \\ &\qquad + \ldots \qquad \ldots\ldots(25), \end{aligned}$$

and

$$\begin{aligned} &\frac{2(1 + q + 2r + 3s + \ldots)(q + 2r + 3s + \ldots)}{(1 + p)q} E(p, q-1, r, s, t \ldots) \\ &\quad = \frac{p(q+1)}{p+1}(p, q+1, r, s, t \ldots) \\ &\qquad + \frac{2(q-1)(r+1)}{q}(p+1, q-1, r+1, s, t \ldots) \\ &\qquad + 3(s+1)(p+1, q, r-1, s+1, t \ldots) \\ &\qquad + 4(t+1)(p+1, q, r, s-1, t+1 \ldots) \\ &\qquad + \ldots \qquad \ldots\ldots(26). \end{aligned}$$

* "Lagrange's Theorem and Stationary Functions," *Trans. Opt. Soc.* **27** 1925–6 (43).

These may be combined to give

$$qD(p,q,r,s,t\ldots)-(q+2r+3s+\ldots)E(p,q-1,r,s,t\ldots)$$
$$=\frac{(1+p)(1+r)}{1+q+2r+3s+\ldots}(p+1,q-1,r+1,s,t\ldots) \qquad \ldots\ldots(27),$$

and
$$\frac{2(1+q+2r+3s+\ldots)}{1+p}[(q+2r+3s+\ldots)E(p,q-1,r,s,t\ldots)$$
$$-(q-1)D(p,q,r,s,t\ldots)]$$
$$=\frac{p(q+1)}{p+1}(p,q+1,r,s,t\ldots)$$
$$+3(s+1)(p+1,q,r-1,s+1,t\ldots)$$
$$+4(t+1)(p+1,q,r,s-1,t+1\ldots)$$
$$+\ldots \qquad \ldots\ldots(28).$$

The expansion can be obtained from these equations without much difficulty. Any values whatsoever may be given to p, q, r, s, t ... so long as the value of $(p, q, r, s, t \ldots)$ is assumed to be zero if any one of the letters represents a negative quantity.

Development by Numerical Law

The preceding sections have shown that the proper form for f_s is a series of terms obtained by applying the operations D and E to v, and that the main difficulty encountered in carrying the expansion to a high order lies in the determination of the numerical coefficients. There is however another difficulty which becomes very apparent when expansion to a high order is attempted—the difficulty of deciding which of the various alternative forms for the expansion should be chosen. Perhaps the simplest course consists in separating these two problems, finding first an expansion which includes all the terms of the new notation, and subsequently eliminating those that are not required.

Just as the coefficients of the first form of the expansion in terms of the x's were found by considering a function of a single variable, so the crude expansion required for the first part of this programme may be obtained from this simple case. Suppose then that initially there is a single variable x, which is to be replaced by a, where $a = \frac{1}{2}x^2$. By (4)

$$\{pq\} = ap_a q_a,$$

so that
$$u = av_a^2,$$

and
$$D = av_a \frac{d}{da}.$$

Also by substitution
$$u\{pq\} = Dp.Dq,$$

and therefore
$$E = \frac{Du}{u} D.$$

Now the coefficients of the original formula are those in the expansion of

$$\frac{1}{(n+2)!}\frac{d^n}{dx^n}(v_x)^{n+2},$$

which in the new variable becomes

$$\frac{2^{n+1}}{(n+2)!}\left(a^{\frac{1}{2}}\frac{d}{da}\right)^n (a^{\frac{1}{2}}v_a)^{n+2},$$

or

$$\frac{2^{n+1}}{(n+2)!}(u^{-\frac{1}{2}}D)^n u^{\frac{n+2}{2}},$$

and this is equal to

$$\frac{1}{n+1}\left(\frac{Du}{u}+\frac{2}{1}D\right)\left(\frac{Du}{u}+\frac{2}{2}D\right)\left(\frac{Du}{u}+\frac{2}{3}D\right)\ldots\left(\frac{Du}{u}+\frac{2}{n-1}D\right)\frac{2}{n}Du \quad \ldots\ldots(29).$$

The problem to be considered is therefore the determination of the numerical coefficients which arise when (29) is expressed in a series of terms involving a succession of operations D and E applied to u. Now obviously

$$\left(\frac{Du}{u}\right)^h Dq = hE\left[\left(\frac{Du}{u}\right)^{h-1} q\right] - (h-1)D\left[\left(\frac{Du}{u}\right)^h q\right] \quad \ldots\ldots(30),$$

and, by substituting for q, the more general identity

$$\frac{\left(\frac{Du}{u}\right)^h}{h!} D^i q = \Sigma E^a\{(1+a-h)D\}^b E^c\{(1+a+c-h)D\}^d \ldots \frac{\left(\frac{Du}{u}\right)^k}{k!} q \quad \ldots\ldots(31)$$

is obtained, where $a, b, c, d \ldots$ take all possible positive integral values (including $a = 0$) which satisfy

$$a+b+c+d+\ldots = i,$$

and where k is either zero or a positive integer satisfying

$$k = h-a-c-\ldots.$$

It appears from (31) that a typical term of (29) may be expressed in the form $\phi(D, E)\, u_r$, where

$$u_r = \left(\frac{Du}{u}\right)^{r-1} Du = \left(\frac{Du}{u}\right)^r u \quad \ldots\ldots(32).$$

By (30) these quantities may be found from the relation

$$u_{r+1} = (1-r)\,Du_r + rEu_{r-1} \quad \ldots\ldots(33),$$

and are thus built up by prefixing D's and E's to quantities already known. By (29) the terms of the next higher multiplicity corresponding to $\phi(D, E)\, u_r$ will be

$$\frac{1}{n+2}\phi(D,E)\left(\frac{Du}{u}\right)^{r-1}\left\{n\frac{Du}{u}+2D\right\}Du,$$

or

$$\frac{1}{n+2}\phi(D,E)\{nu_{r+1}+2(2-r)\,Du_r+2(r-1)\,Eu_{r-1}\} \quad \ldots\ldots(34),$$

where it will be noted that the new operations are performed before the previously existing operations represented by $\phi(D, E)$. If the complete expansion is required up to a certain point, (34) gives a very simple means of proceeding from one order to the next. For instance, the expansion carried to the stage at which the eleventh powers of the coefficients of f are retained, which includes all terms having powers

of x less than the 26th, and in the optical application corresponds to the retention of the 451 monorythmic aberrations of the 11th and lower orders, is in this notation

$$
\begin{aligned}
f_s = f &+ u + u_1 + \tfrac{1}{8}(u_3 + 2Du_2) + \tfrac{1}{8}(u_3 + 2Du_2 + Eu_1 + 2D^2u_1) \\
&+ \tfrac{1}{240}(3u_4 + 4Du_3 + 7Eu_2 + 6D^2u_2 + 2EDu_1 + 4DEu_1 + 4D^2u_1) \\
&+ \tfrac{1}{960}(6u_5 + 2Du_4 + 23Eu_3 + 8D^2u_3 + 4EDu_2 + 16DEu_2 + 8D^3u_2 + 7E^2u_1 \\
&\quad + 2ED^2u_1 + 4DEDu_1 + 6D^2Eu_1 + 4D^4u_1) \\
&+ \tfrac{1}{8640}(30u_6 - 26Du_5 + 163Eu_4 + 32D^2u_4 - 26EDu_3 + 92DEu_3 + 24D^3u_3 \\
&\quad + 127E^2u_2 + 10ED^2u_2 + 20DEDu_2 + 62D^2Eu_2 + 20D^4u_2 + 14E^2Du_1 + 8EDEu_1 \\
&\quad + 32DE^2u_1 + 4ED^3u_1 + 8DED^2u_1 + 12D^2EDu_1 + 16D^3Eu_1 + 8D^5u_1) \\
&+ \tfrac{1}{362880}(90u_7 - 198Du_6 + 639Eu_5 + 174D^2u_5 - 404EDu_4 + 172DEu_4 + 8D^3u_4 \\
&\quad + 870E^2u_3 + 56ED^2u_3 - 32DEDu_3 + 282D^2Eu_3 + 36D^4u_3 + 42E^2Du_2 \\
&\quad - 28EDEu_2 + 280DE^2u_2 + 12ED^3u_2 + 24DED^2u_2 + 36D^3Eu_2 + 96D^5Eu_2 \\
&\quad + 24D^5u_2 + 127E^3u_1 + 14E^2D^2u_1 + 8EDEDu_1 + 10ED^2Eu_1 + 32DE^2Du_1 \\
&\quad + 20DEDEu_1 + 62D^2E^2u_1 + 4ED^4u_1 + 8DED^3u_1 + 12D^2ED^2u_1 + 16D^3EDu_1 \\
&\quad + 20D^4Eu_1 + 8D^6u_1) \\
&+ \tfrac{1}{3628800}(630u_8 - 2286Du_7 + 5553Eu_6 + 2802D^2u_6 - 6662EDu_5 - 776DEu_5 \\
&\quad - 988D^2u_5 + 11202E^2u_4 + 2008ED^2u_4 - 912DEDu_4 + 3366D^2Eu_4 + 220D^4u_4 \\
&\quad - 1446E^2Du_3 - 2620EDEu_3 + 2992DE^2u_3 - 28ED^3u_3 + 232DED^2u_3 \\
&\quad - 312D^2EDu_3 + 720D^3Eu_3 + 96D^5u_3 + 4369E^3u_2 + 98E^2D^2u_2 + 56EDEDu_2 \\
&\quad + 294ED^2Eu_2 + 224DE^2Du_2 + 12DEDEu_2 + 1562D^2E^2u_2 + 28ED^4u_2 \\
&\quad + 56DED^3u_2 + 84D^2ED^2u_2 + 112D^3EDu_2 + 284D^4Eu_2 + 56D^6u_2 + 254E^2Du_1 \\
&\quad + 84E^2DEu_1 - 56EDE^2u_1 + 560DE^3u_1 + 28E^2D^3u_1 + 16EDED^2u_1 \\
&\quad + 20ED^2EDu_1 + 24ED^3Eu_1 + 64DE^2D^2u_1 + 40DEDEDu_1 + 48DED^2Eu_1 \\
&\quad + 124D^2E^2Du_1 + 72D^2EDEu_1 + 192D^3E^2u_1 + 8ED^5u_1 + 16DED^4u_1 \\
&\quad + 24D^3ED^2u_1 + 32D^3ED^2u_1 + 40D^4EDu_1 + 48D^5Eu_1 + 16D^7u_1) \\
&+ \tfrac{1}{11184000}(2520u_9 - 12924Du_8 + 26622Eu_7 + 22638D^2u_7 - 48860EDu_6 \\
&\quad - 16820DEu_6 - 15160D^2u_6 + 72573E^2u_5 + 28018ED^2u_5 - 1320DEDu_5 \\
&\quad + 27474D^2Eu_5 + 3844D^4u_5 - 28188E^2Du_4 - 37128EDEu_4 + 8864DE^2u_4 \\
&\quad - 4128ED^3u_4 + 2752DED^2u_4 - 7980D^2EDu_4 - 1072D^3Eu_4 - 56D^5u_4 \\
&\quad + 51082E^3u_3 + 1838E^2D^2u_3 + 2844EDEDu_3 + 7200ED^2Eu_3 - 2096DE^2Du_3 \\
&\quad - 2688DEDEu_3 + 16346D^2E^2u_3 + 140ED^4u_3 - 8DED^3u_3 + 648D^2ED^2u_3 \\
&\quad - 272D^3EDu_3 + 1796D^4Eu_3 + 128D^6u_3 + 1016E^2Du_2 - 2556E^2DEu_2 \\
&\quad - 5464EDE^2u_2 + 8224DE^2u_2 + 112E^2D^2u_2 + 64EDED^2u_2 + 80ED^2EDu_2 \\
&\quad + 40ED^3Eu_2 + 256DE^2D^2u_2 + 160DEDEDu_2 + 656DED^2Eu_2 + 496D^2E^2Du_2 \\
&\quad - 336D^2EDEu_2 + 2208D^3E^2u_2 + 32ED^5u_2 + 64DED^4u_2 + 96D^3ED^2u_2 \\
&\quad + 128D^3ED^2u_2 + 160D^4EDu_2 + 384D^5Eu_2 + 64D^7u_2 + 4369E^4u_1 + 254E^3D^2u_1 \\
&\quad + 84E^3DEDu_1 + 98E^2D^2Eu_1 - 56EDE^2Du_1 + 56EDEDEu_1 + 294ED^2E^2u_1 \\
&\quad + 560DE^3Du_1 + 224DE^2DEu_1 + 12DEDE^2u_1 + 1562D^2E^3u_1 + 28E^2D^4u_1 \\
&\quad + 16EDED^3u_1 + 20ED^3ED^2u_1 + 24ED^3EDu_1 + 28ED^4Eu_1 + 64DE^2D^3u_1 \\
&\quad + 40DEDED^2u_1 + 48DED^2EDu_1 + 56DED^3Eu_1 + 124D^3E^2D^2u_1 \\
&\quad + 72D^3EDEDu_1 + 84D^3ED^2Eu_1 + 192D^3E^2Du_1 + 112D^3EDEu_1 + 284D^4E^2u_1 \\
&\quad + 8ED^6u_1 + 16DED^5u_1 + 24D^2ED^4u_1 + 32D^3ED^3u_1 + 40D^4ED^2u_1 \\
&\quad + 48D^5EDu_1 + 56D^6Eu_1 + 16D^8u_1) \\
&+ \ldots \qquad \ldots\ldots(35).
\end{aligned}
$$

After the expansion has been obtained in this form—and not before—the number of terms may be reduced by expressing all the operations explicitly in terms of D and E by the aid of (33). This simplified expansion is not however in its lowest terms, for with the exception of the term $DEDu$ every possible combination of the letters D and E is found operating on u. The superfluous terms have now to be removed by the use of identities such as equation (19).

Identical Relations

Since every term $\phi(D, E)\,u$ is expressible separately in terms of the original sums involving the x's, the number of terms of any multiplicity in the new notation must be reducible to at least as few as in the original expansion (1). That is to say, a large number of identical relations must exist. These relations may, of course, be found by the method adopted in deriving the expansion in the first part of this paper, retaining q as the variable throughout. For instance, this leads to the identity

$$(E^2D - 3EDE + 2DE^2)\,Dq + (D^2E - 3DED + 2ED^2)\,Eq = 0,$$

which may also be written in the form

$$(E^2D - 3EDE + 2DE^2)\,q + (D^2E - 3DED + 2ED^2)\frac{Du}{u}\,q = 0 \qquad \ldots\ldots(36),$$

a relation of which (19) is evidently only the simplest case. By prefixing $\phi(D, E)$ to this identity and substituting $\psi(D, E)\,q$ for q a very large number of relations is obtained. These are found to be insufficient to reduce the number of terms to the extent required. Other relations may be found by carrying the original procedure further, and the next two general relations to be used later have in fact been verified in this way. It is preferable however to obtain an equation which includes all the identities of this type which exist.

In deriving these relations we shall make use of the equation

$$E = \frac{Du}{u}\,D$$

found in the previous section when the function contains only a single variable. It may be noted that by the definition (21), however many variables are present,

$$u = Dv,$$

$$Ev = \{uv\} = \{vu\} = Du$$

$$= \frac{Du}{u}\,Dv,$$

so that when operating on v the relation is always true. When operating on a general function this no longer holds, but as in other sections we may nevertheless construct relations on simple cases with the assurance that they will hold generally. This may be traced to the fact that we are at liberty in any term of the expansion to transfer any operations from a given factor to v itself. Alternatively, it may be

justified by noting that q undergoes the same operations in both Dq and Eq, the sole difference being in the coefficients.

The relations to be established may be based upon the identity

$$p_{j+1}\left[\frac{d^{i+1}}{dx^{i+1}}(pq)-\frac{d}{dx}\sum_{t=1}^{i}\binom{i}{t}p_{i-t}q_t\right]+p_j\frac{d}{dx}(p_iq_1)$$
$$-p_{i+1}\left[\frac{d^{j+1}}{dx^{j+1}}(pq)-\frac{d}{dx}\sum_{t=1}^{j}\binom{j}{t}p_{j-t}q_t\right]-p_i\frac{d}{dx}(p_jq_1)=0,$$

which readily follows from

$$\frac{d^{i+1}}{dx^{i+1}}(pq)-\frac{d}{dx}\sum_{t=0}^{i}\binom{i}{t}p_{i-t}q_t=0.$$

In this formula $\frac{d}{dx}$ and p are to be replaced by D and $\frac{Du}{u}$, which, when E is defined as $\frac{Du}{u}D$, involves the substitutions

$$p_iD=\sum_{0}^{i}\binom{i}{t}D^{i-t}E(-D)^t,$$

and

$$\sum_{1}^{i}\binom{i}{t}p_{i-t}D^t=\sum_{1}^{i}\binom{i}{t}D^{i-t}E(-D)^{t-1}.$$

The identity may therefore be written

$$\Sigma\left[\binom{i+1}{t}-(-)^{i+j}\binom{j+1}{t}\right]D^tE(-D)^{i+j+1-t}\left(\frac{Du}{u}q\right)$$
$$+\Sigma\left[\binom{i+1}{r}\binom{j+1}{s+1}-\binom{i}{r-1}\binom{j}{s}-(-)^{i+j}\left\{\binom{j+1}{r}\binom{i+1}{s+1}-\binom{j}{r-1}\binom{i}{s}\right\}\right]$$
$$\times D^rE(-D)^{i+j-r-s}ED^sq=0.$$

It is unnecessary to consider any other case than $i\sim j=1$, for all other identical relations are derivable by differentiation. The formula then becomes

$$i!\,\Sigma\frac{2i-r-s+2}{(i-r+1)!(i-s+1)!}\left[\frac{(i+1)!}{r!s!}-\frac{(i-1)!}{(r-1)!(s-1)!}\right]D^rE(-D)^{2i-r-s}ED^{s-1}q=0,$$

where $ED^{-1}q$ is to be interpreted as $\frac{Du}{u}q$. This expression can be further simplified, for several terms can be removed by the aid of the earlier relations of the same series. By writing $i-k$, $r-k$, $s-k$ for i, r, s respectively, replacing q by D^kq, and operating on the result by D^k we obtain

$$(i-k)!\,\Sigma\frac{2i-r-s+2}{(i-r+1)!(i-s+1)!}$$
$$\left[\frac{(i-k+1)!}{(r-k)!(s-k)!}-\frac{(i-k-1)!}{(r-k-1)!(s-k-1)!}\right]D^rE(-D)^{2i-r-s}ED^{s-1}q=0.$$

Multiply this by $(-)^k\binom{i-1}{k}$ and sum with respect to k. The result is

$$i!\,\Sigma\frac{2i-r-s+2}{(i-r-s+1)!r!s!}D^rE(-D)^{2i-r-s}ED^{s-1}q=0 \quad \ldots\ldots(37),$$

from which it is apparent that the lowest power of D separating the two E's is D^{i-1}. This characteristic also shows that all the relations of this series are independent.

The simplification of (35) requires the use of (36), which corresponds to $i = 1$, and the equations for $i = 2$ and $i = 3$, viz.

$$(EDED^2 - 4ED^2ED + 5ED^2E + 3DEDED - 8DED^2E + 3D^2EDE)\,q + (D^2E - 4D^2ED + 5DED^2 - 2ED^3)\,D\left(\frac{Du}{u}q\right) = 0 \quad \ldots\ldots(38),$$

and

$$(ED^2ED^2 - 5ED^2ED^2 + 9ED^4ED - 7ED^5E + 4DED^2ED^2 - 15DED^3ED + 18DED^4E + 6D^2ED^2ED - 15D^2ED^3E + 4D^3ED^2E)\,q + (D^4E - 5D^3ED + 9D^2ED^2 - 7DED^3 + 2ED^4)\,D^2\left(\frac{Du}{u}q\right) = 0 \ldots\ldots(39).$$

No other series of relations exists. Should there, for instance, be any distinct relations involving three E's, they would be derived from an identity containing three p's. The various types of relation of the necessary form can be reduced to (37). Examples are forms reducible to the case when q contains p as a factor; identities such as

$$(D^2E - 3DED + 2ED^2)\left\{2D^3\left(\frac{Du}{u}\right)^2 + (2ED^2 - 3DED - 3D^2E)\frac{Du}{u} + 6DE^2 - 4EDE\right\}q - (E^2D - 3EDE + 2DE^2)\,(D^2E - 3DED + 2ED^2)\,q = 0,$$

which, if $\frac{Du}{u}q$ is written for q, may be put into the form

$$(E^2D - 3EDE + 2DE^2)\,(D^2E - 3DED + 2ED^2)\frac{Du}{u}q - (D^2E - 3DED + 2ED^2)\frac{Du}{u}(E^2D - 3EDE + 2DE^2)\,q = 0,$$

and is thus a consequence of (36); and equations of such forms as

$$p_aDp_bDf_c + p_bDp_cDf_a + p_cDp_aDf_b - p_bDp_aDf_c - p_cDp_bDf_a - p_aDp_cDf_b = 0,$$

where

$$f_t = D^t(pq) - \Sigma\binom{t}{n}p_{t-n}q_n,$$

which may be compounded of three such as have already been considered. We conclude then that the expansion will be in its simplest form when it has been reduced as far as possible by (37).

The number of terms present at various stages in the reduction is given in the accompanying table. Column (1) gives the value of h for the group of terms ending with $\frac{2^h}{(h+1)!}D^hu$: column (2) the number of terms belonging to this group in equation (35); each new entry exceeds by unity the sum of the two previous numbers: column (3), in which each term is the sum of the two preceding, the number of terms remaining when (35) has been reduced by (33): columns (4) to (8) the number

of relations derived from the first five of equations (37): column (9) is the excess of the entry under column (3) over the sum of the numbers entered in columns (4) to (8), and represents an upper limit to the number of terms which may be present in the final formula: column (10) gives the number of partitions of h, that is, the number of terms in formula (1), which also constitutes an upper limit to the number of terms to be permitted in the final formula: column (11) gives the actual number of terms present in this formula as given later.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
4	7	5	—	—	—	—	—	5	5	4
5	12	8	1	—	—	—	—	7	7	7
6	20	13	2	—	—	—	—	11	11	10
7	33	21	5	1	—	—	—	15	15	15
8	54	34	10	2	—	—	—	22	22	21
9	88	55	20	5	1	—	—	29	30	29
10	143	89	38 − 1	10	2	—	—	40	42	
11	232	144	71 − 3	20	5	1	—	50	54	
12	376	233	130 − 9	38 − 2	10	2	—	64	77	
13	609	377	235 − 22	71 − 6	20	5	1	73	101	
14	986	610	420 − 51	130 − 19	38 − 2	10	2	82	135	

The fact that no number in column (9) exceeds that in (10) tends in itself to show that no other series of relations exists. Any additional relations would be likely to cause the series to terminate. It will be noticed that for even values of h the actual number of terms in the final formula is less by unity than the upper limit given in column (9).

The positive entries in column (4) may be built up by differences, the successive orders starting with the number of their order repeated. Thus

0		1		2		5		10		20		38		71
	1		1		3		5		10		18		33	
				2		2		5		8		15		
							3		3		7		11	
										4		4		9
												5		5
													6	

Their origin is of course that column (3) represents $\phi(h)$, where

$$\phi(h) = \Sigma \binom{h-r}{r},$$

and the positive entries in (4) represent

$$\Sigma\phi(r)\,\phi(h-r).$$

The deductions from these values arise from the repetition of the same function in this enumeration. For example, from the identity $\theta q = 0$ we derive $\phi\theta\psi u = 0$, where ϕ and ψ are arbitrary functions of D and E. Now if either ϕ or ψ or both contain terms which may be built up into a function like θ which itself vanishes, it is clear that the same relation has been counted twice at least. The number of deductions to be made is readily seen when their origin is appreciated.

It may be noted that the existence of the whole series of relations (37) cannot be deduced from the number of terms to be removed. The first and second are

clearly necessary; the third could only be inferred from the one line $h = 11$; and the rest could not be supported at all in this way.

The Final Formula

All the material is now provided for the construction of the final formula; the remainder is a mere matter of arithmetic. There is, however, one feature in this work which is less simple than might be thought—uncertainty which of the terms should be removed. Of course the choice is in one sense of no importance, since all the various expressions which may be constructed are algebraically equivalent to one another. Whatever terms may be eliminated at one stage should determine some of the terms to be removed at all subsequent stages, otherwise the saving of labour in applications which is the prime reason for reducing the formula to its lowest terms is lost. Moreover, it is very desirable to adopt that form which yields the simplest numerical coefficients. Unfortunately this condition can only be met by finding the coefficients resulting from the removal of various combinations of terms. Nor is it always the case that a selection which gives very simple coefficients for a given group of terms will be consistent with simple coefficients in terms of higher multiplicity. Guidance might be obtained from algebraic expressions for the coefficients in equation (35), but it does not appear likely that the gain would repay the trouble involved. It cannot therefore be said that the formula quoted hereafter is necessarily the best that might be obtained.

There are a few features of the reduction to which attention may be called. It has already been seen that any term of the form $\phi(D, E)\, DEDu$ should be eliminated. These form a particular group of the more general class represented by $\phi(D, E)\, DE^iDu$, which have all been removed. There can be no doubt that these terms should be deleted from the final formula; for if relations derived from (36) are used to eliminate from the terms of the same order as E^nu or $D^{2n}u$ all terms of the form $\phi(D, E)\, DE^iDu$ for every value of i less than $n-1$, it will be found that the coefficient of $DE^{n-1}Du$ in the reduced expression is zero. This is the reason why the number of terms indicated in column (11) of the table is less by unity than the maximum number given in column (9) for even values of the key integer in column (1).

After the terms just mentioned have been removed, it will be found that all the numerical coefficients for orders containing a term E^nu are even, but in the alternative orders, where the leading term is of the form E^nDu, at least two odd coefficients occur. It follows that in the former group a factor 2 can be removed from the common denominator, but in the latter group no such reduction is possible.

In the final formula an attempt has been made to select on a systematic basis the additional terms for deletion. The expression, obtained after somewhat heavy arithmetical work, is

$$\begin{aligned} f_s = f &+ u + Du + \tfrac{1}{3}(E + 2D^2)\,u + \tfrac{1}{8}(3ED + DE + 2D^3)\,u \\ &+ \tfrac{1}{15}(8E^2 + ED^2 + 4D^2E + 2D^4)\,u \\ &+ \tfrac{1}{960}(15E^2D + 215EDE - 204DE^2 - 170ED^3 + 262DED^2 - 32D^3E + 4D^5)\,u \end{aligned}$$

$$+ \tfrac{1}{315}(533E^3 - 209E^2D^2 - 101ED^2E - 1820DEDE + 2336D^2E^2 + 2ED^4 + 1820DED^3 - 2718D^3ED^2 + 468D^4E + 4D^6)u$$

$$+ \tfrac{1}{2835}(105E^3D + 49875EDE^2 - 65000DE^3 - 49210EDED^2 - 30240ED^3E + 66880DE^2D^2 + 64310DED^2E + 3150D^3EDE - 69178D^3E^2 + 31360ED^5 - 78382DED^4 - 3780D^2ED^3 + 84094D^3ED^2 - 16592D^5E + 8D^7)u$$

$$+ \tfrac{1}{113400}(93710E^4 - 71381E^2D^2 - 27999E^2D^2E - 26180EDEDE - 1202124DEDE^2 + 1732072D^2E^3 + 20952E^2D^4 + 26180EDED^3 - 21748ED^2ED^2 + 15282ED^4E + 1362592DEDED^2 + 640948DED^3E - 1962058D^2E^2D^2 - 1783542D^3ED^2E + 1615236D^4E^2 + 4ED^6 - 801416DED^5 + 2003572D^2ED^4 + 2003516D^4ED^2 + 400748D^6E + 8D^8)u$$

$$+ \tfrac{1}{113400}(945E^4D + 98107159251E^2DE^2 - 2943019019028EDE^3 + 19618515965 2DE^4 - 152727430460E^2DED^2 + 108572597092E^2D^3E + 54474353062EDE^2D^2 - 329253102266EDED^3E + 14289864826ED^2EDE + 3129868034ED^3E^2 - 363171029482DE^2D^2 + 217270360494DE^2D^2E - 10717230532DEDEDE - 3302032396DED^3E^2 - 136662399476ED^2ED^3 + 201941211462ED^3ED^2 - 32784559516ED^5E + 34028413570DE^2D^4 + 102496624432DEDED^3 - 247451409556DED^2ED^2 + 50864136506DED^4E + 16831452876D^2EDED^2 - 181143032 78D^2ED^3E - 1004392936ED^7 + 351537532 0DED^6 - 42184503 84D^2ED^5 + 17576877 92D^3ED^4 - 25109720D^7E + 16D^9)u$$

$$+ \ldots \qquad \ldots\ldots(40).$$

It may be found useful to add a note, based on the experience gained in deriving this formula, on the simplification of algebraic expressions by the use of several identities connecting the terms. When it is known which terms are to be removed, the problem is simple. The determinant, having as elements the coefficients in the identities of the terms to be removed, is evaluated, together with all its first minors. If there are several conditions labour will be saved by employing a method of calculation described by the author*. From these numbers the factors by which the various identities are to be multiplied before being added to the simplificand are easily found. The same procedure should be followed as a preliminary step to remove any terms which it is known will have to be deleted. For example, after

* "The Calculation of Determinants and their Minors," *Phil. Mag.* 3 1927 (1007).

reducing (35) by (33), 55 terms remained in the last bracket, and 26 identities were available for further simplification. The first step was to remove the 12 terms of the form $\phi(D, E)\, DE^{i}Du$ from the simplificand and from the other 14 identities. There remained the problem of selecting 14 terms out of about 40 eligible for elimination. The possible number of combinations is extremely great.

As a rule some combinations will be inadmissible since the conditions are mutually inconsistent, and others may, as in this formula, be arbitrarily rejected because they involve the use of fractional multipliers for the identities. The former condition corresponds to the vanishing of the determinant of the coefficients of the selected terms in the identities, and the latter result will almost invariably follow unless the value of the determinant is a very simple number. It should be possible to limit the field of choice at the outset. Thus with 14 conditions, after a few simple trials (in which it may easily be discovered that the appearance of simplicity is apt to be deceptive) it should be possible to select, say, 10 terms which are not unlikely to belong to a group satisfying the conditions. Even so the labour involved in ringing the changes on the remaining terms, to say nothing of rejecting any of the ten which fail to meet expectations, may prove prohibitive unless it is done in the simplest way.

The method which has proved most satisfactory is to evaluate the determinant and all its minors for the terms judged *a priori* to be best suited for removal. If the determinant vanishes, one or more of the terms must be changed until a non-vanishing determinant, of the same order as the number of identities, is obtained. It will probably be found that the selected combination does not comply with all the desired conditions, so that further changes in the selected terms must be made. The new determinant and its minors, for any proposed substitution, can be found very simply from the old values. Let Δ be the value of the old determinant, and A_{rs} the minor of the element in the rth row and sth column. Suppose the elements of the pth row of the original determinant are to be replaced by $b_1, b_2, \ldots b_n$. We first find the n quantities

$$\sigma_r = \sum_{s=1}^{n} A_{rs} b_s \qquad (r = 1, 2, \ldots n).$$

If we denote the new minors by the letter B we evidently have

$$B_{ps} = A_{ps},$$

and the value of the new determinant is σ_p. The remaining minors are found from the equation

$$\Delta B_{rs} = \sigma_p A_{rs} - \sigma_r A_{ps} \qquad (r \neq p),$$

which may be verified readily from the fundamental relations between the elements and the minors of any determinant. The evaluation of the σ's will show at once whether the substitution of any proposed new term is likely to prove profitable, and in conjunction with the known minors they are of assistance in deciding which of the existing terms should be displaced. Several terms can be replaced by repeating this process.

REVIEW

Proceedings of the Optical Convention, 1926. Part I, 512 pp.; Part II, 578 pp. Fully illustrated with plates and figures in the text. (London: The Optical Convention, 1, Lowther Gardens, Exhibition Road, S.W. 7.) Price £3 net. Inland postage 1*s.* 3*d.* extra.

Part II* of the *Proceedings* of the Optical Convention (1926) contains more than fifty papers of a most interesting and varied character, dealing with historical instruments, surveying instruments, spectrometers, lens theory, optical projection, cinematography, the manufacture of microscopes and lenses, and other subjects.

The volume commences with four papers on historical instruments: the first by Dr Clay on "A brief account of some of the historical instruments exhibited at the Optical Convention"; the second by Mr R. S. Whipple on "An old catalogue and what it tells us of the scientific instruments and curios collected by Queen Charlotte and King George III"; and the third and fourth by Messrs D. Baxandall and T. H. Court on (1) "A telescope made by Christopher Cock in 1673" and (2) "Early optical and other scientific instruments as documents of historical value." To appreciate the value of these historical papers the early history of optical instruments, at least in outline, should be remembered. Galileo invented the telescope, of which all the telescopes of to-day are probably lineal descendants, in the year 1608. Up to that time it may be said that the greatest optical achievement of ten thousand years of continuous civilisation had been—a pair of spectacles! Old-age sight (presbyopia) had been corrected about the year 1300, and short sight two centuries later. Simple mirrors, both plane and curved, were known much earlier, but it is a remarkable fact that lenses as picture makers were apparently quite unknown to the ancient civilisations of Assyria, Babylonia, and Egypt. The Roman egg merchant, however, we are told, used to expose his eggs for sale submerged in water in a glass globe to make them look "fine and large." Another interesting fact is that, although the telescope was not invented until 1608, lensless telescopic systems in effect had been employed by astronomers for many years before for observing eclipses of the Sun. An image of the Sun was projected through a "pinhole" on to a ruled screen (wall) in a darkened room and, since the observer was at a less distance from the image on the screen than the pinhole was, he saw that image magnified correspondingly. A lens was later employed to allow of the use of a larger pinhole without loss of definition and, later still, as used by Jeremiah Horrox in 1639, a Galilean telescope replaced the pinhole.

Modern optical history, apart from that of spectacles, may therefore be looked upon as dating from the beginning of the seventeenth century. Extraordinary progress was made in the century and a half which elapsed between the invention of the simple telescope and the achromatic telescope by Dollond in 1758. The papers in the *Proceedings* to which we are referring derive a peculiar interest from the fact that they throw light upon the optical achievements and instruments of the latter part of this period.

Dr Clay in his admirable paper draws particular attention to (1) the actual microscope that was used by Dr Hooke and described in his *Micrographia* (1665); (2) two microscopes by Campani (*fl.* 1660); (3) A number of Lieberkuhn's simple single-lens microscopes; and (4) in later times famous objectives by Tolles, and Ross. Messrs Baxandall and Court give a detailed account of an old telescope of the three-lens erecting eyepiece type made by Cock in 1673. It is the best description of an old telescope we have ever seen and therefore to be recommended to the student. Another quaint and invaluable inclusion in the paper is a reproduction of Edward Scarlett's Trade-card (*c.* 1728), which

* For review of Part I see *Trans. Opt. Soc.* **28** 1926–7 (168).

shows clearly what optical instruments were available at that date. Everyone interested in the history of optical instruments will be relieved to hear what Messrs Baxandall and Court have to say about the famous collection of old microscopes which was made by Sir Frank Crisp, assisted by Mr Mayell, during the latter half of the last century. After Sir Frank's death in 1919 it was unfortunately found that he had neglected to take the necessary legal steps for the carrying out of his expressed decision to bequeath his collection to the Science Museum. "Since that date great efforts have been made to acquire the whole of the collection for the nation. The executors, however, decided to sell by public auction, and at the sales in 1920, 1921, and 1925 the collection was disposed of. The Science Museum purchased about fifty examples, and Mr Court purchased more than half of the whole collection." It is unlikely that Mr Court has selected the less interesting half of the collection.

Mr R. S. Whipple deals with a hitherto unpublished catalogue of instruments and curios preserved in the private observatory of King George III, built at Richmond in 1769. The collection described was dispersed in 1841. The paper constitutes an invaluable record but it must be read in detail.

The papers on Surveying instruments and methods include amongst others the following: "Notes on survey instruments" by the Survey of India; "Photographic rectification" by Lieut.-Colonel MacLeod; "Levelling instruments" by Major Henrici; "Optical Methods of distance measurement" by Mr J. Martin Strang; "Sextants by mass production" by Instructor-Commander Baker and "Air photography apparatus" by Mr H. B. Stringer.

A number of papers on the theory of lenses are introduced by a charming one on "A review of lens theory" by Mr T. Smith, who in this case makes one of his rare descents from Olympus and appeals to mortals.

Amongst papers of interest to the lens designer and computer may be mentioned "The Taylor-Hobson F/2.5 anastigmat lens" and "The development of the telephotographic objective" by Mr H. W. Lee; also "The telephoto lens" by Mr L. B. Booth.

A description of the spectrometer built for the National Physical Laboratory, Teddington, by Messrs Watts is given by Mr J. Guild.

Graphical methods of optical designing are dealt with by Mr J. H. Dowell.

Professor Rankine has something to say on the "Optical problems relating to 'speaking films'" whilst Professor Pollard describes an interesting example of geometrical designing in "An eyepiece and interchangeable nosepiece for 'centring' and 'squaring on' microscope objectives."

One of the last papers in the volume is a suggestive one by Mr Conrad Beck on "Some recent advances in the manufacture and use of the microscope." The importance and advances realised by modern methods of high-power illumination are indicated, whilst the value of Mr J. E. Barnard's interferometer method of testing the microscope stand is insisted upon.

Students interested in instrument design will find food for thought in a paper on the "Universal lens interferometer" by Mr J. H. Dowell of Messrs Adam Hilger, Ltd.

A large number of papers cannot be named even, but enough has perhaps been said to indicate the value of these two volumes of the *Proceedings* of the Optical Convention (1926) to the student of optics.

F. J. C.

ABSTRACTS OF PATENT SPECIFICATIONS

264810*. Zeiss, C. REFLECTORS.

In a bell-shaped reflector, the transition from each zone to the next is made by a series of inward steps, so reducing the aperture. The step portions constitute spherical surfaces struck from the light source as centre.

264842*. Junkers, H. NAVIGATING AIRCRAFT.

Images of the compass card and of the landscape are superposed or thrown upon screens placed side by side, and the observed direction of drift of the ground is carried over to a pointer which indicates the course to be steered.

265069. Berthon, R. PROJECTION APPARATUS.

In order that the linear effect of a trichrome screen may not be apparent in a projected image, the beam is divided into three zones at the objective. One portion passes direct, while the others are diverted by passage through prisms placed with their edges on opposite sides, and of such an angle that the light is shifted upon the screen by an amount equal to the width of one line.

265770. Holophane, Ltd., and another. PHOTOMETRIC APPARATUS.

In a comparison photometer, the test surface is mounted upon a frame which carries also a graduated semicircle upon which the inclination of the surface to the horizontal can be read off by means of a weighted pointer. A sighting-tube, with lens and cross-wires, measures the angle of incidence of the light.

266362*. Ryan, W. D. VEHICLE LAMPS.

The reflector is of substantially conoidal form, with the upper and lower portions pressed in towards the axis to produce a beam with a convex top, increasing in brilliancy towards the centre. The front glass has prisms formed upon the sides and lower part to distribute light laterally.

266564. Television, Ltd., and another. TELEVISION.

Discs having two spiral series of holes are employed in conjunction with separate sensitive cells or sources of light. Each series may be employed to explore or reconstitute a portion of the image, or may be used in connection with one of a pair of stereoscopic images.

266767. Culver, G., Ltd., and another. CUTTING OUT LENSES.

The lens is held upon a table which is rotated through hand gearing. The cutter is guided to cut the lens to the shape desired, and means are also provided to ensure that the correct angle of cut corresponding to the curvature is maintained.

266869. Berry, H. H. STEREOSCOPES.

A portable stereoscope has the lens front hinged to the base, and carries a number of slides hinged to fold book-wise upon two-piece loops or rings. The back is held when open by a stop hinge.

* Specification published before acceptance.

267133*. I. G. Farbenindustrie Akt. Ges. LENSES.

An objective consisting of two exterior converging components and an intermediate diverging component is associated with interchangeable components to give a new combination having a greater focal length, while maintaining the back focus substantially unchanged.

267171. Steele, L. J., and another. PROJECTORS.

A hemispherical reflector is continued into a truncated ellipsoidal reflector, and the source of light is placed at a point which is both the centre of the hemisphere and the focus of the ellipsoid. Light which does not meet the reflector passes through a refracting element which converges it to the conjugate focus of the ellipsoid. The convergence of the final beam may be modified by an additional lens.

267594. Hilger, A., Ltd., and another. STEREOSCOPES.

The instrument is designed to provide for the viewing of stereoscopic prints of various sizes, taken with lenses of various focal lengths. The eye tubes consist of telescopes of ordinary astronomical type fitted with erecting prisms. The prints are supported in holders which are adjustable for position and tilt, and are viewed by reflection.

267891*. Wernicke & Co. EDGING LENSES.

The lens is held between pads upon spindles receiving rotation from the grinding disc, and is pressed against the latter by a hand lever. The carrying frame is set to give the required bevel, as shown upon a dividing disc.

268470. Hanson, E. T., and another. RANGE-FINDING.

Instructional apparatus for use in a limited space comprises, in addition to a range or height finder, a collimating device, which may be formed by an ordinary range-finder having the centre combination replaced by a half-silvered prism. Light passes from the collimator into the range-finder, which is mounted upon a pedestal providing an independent movement in azimuth.

268534. Hughes, W. VEHICLE LAMPS.

An anti-dazzle fitting in the form of a unit adapted for attachment to ordinary lamps consists of a bowl-shaped glass front having within it a series of concentric tubular or frustum shaped partitions, which may be of glass—clear, tinted, or obscured—or of metal—perforated, polished, or otherwise treated.

* Specification published before acceptance.

R. J. T.

13th January, 1927.

Meeting held at the Imperial College of Science.

Mr T. SMITH, M.A., President, in the Chair.

Elections: (Membership) Mr F. POCOCK and K. SUNAGAMA.
(Corporate Membership) Messrs GEORGE CULVER, Ltd.
(Student Membership) Mr J. R. HAMBLIN.

The following papers were read:

"Contributions to the Theory of Stereoscopic Vision." By Instructor Lieut.-Commander N. M. S. LANGLANDS, R.N., M.A., B.Sc.

"The Theory of the Lens-testing Interferometer." Part I. By T. SMITH, M.A., F.Inst.P.

10th February, 1927.

Meeting held at the Imperial College of Science.

Mr T. SMITH, M.A., President, in the Chair.

Election: (Membership) Mr O. P. RAPHAEL.

Mr Smith delivered his Presidential Address entitled "Some Uncultivated Optical Fields."

10th February, 1927.

Annual General Meeting held at the Imperial College of Science.

Mr T. SMITH, M.A., President, in the Chair.

The Hon. Secretary presented the Annual Report* of the Council, which was approved.

The Treasurer's Report† and Balance Sheet‡ were presented by the Hon. Secretary, in the absence of the Hon. Treasurer, and were approved.

The Chairman read the list of persons nominated by the Council for office in the year 1927–28 and declared all these duly elected as no other nominations had been received:

President: Dr R. S. CLAY.
Vice-Presidents: Mr F. F. S. BRYSON, Mr H. H. EMSLEY, Mr J. GUILD, Mr F. C. WATTS.
Business Secretary: Prof. A. F. C. POLLARD.
Papers Secretary: Mr W. B. COUTTS.
Treasurer: Major E. O. HENRICI.
Librarian: Mr J. H. SUTCLIFFE.
Editor: Dr JOHN S. ANDERSON.

Council:

Already serving:	Elected, February, 1927:
Mr D. BAXANDALL.	Instr. Lieut.-Com. N. M. S. LANGLANDS.
Mr W. M. BRETT.	Prof. A. O. RANKINE.
Mr E. F. FINCHAM.	Mr R. S. WHIPPLE.
Mr E. T. HANSON.	Col. H. S. L. WINTERBOTHAM.
Mr H. C. RAXWORTHY.	
Mr J. RHEINBERG.	
Dr G. F. C. SEARLE.	
Mr W. SWAINE.	

Votes of thanks to the retiring Officers and Members of Council, to the Hon. Auditors, and to the Governors of the Imperial College, with Prof. Callendar, for the use of the College premises for the meetings of the Society were unanimously passed.

* See p. vii. † See p. x. ‡ See p. xii.

10th March, 1927.

Meeting held at the Imperial College of Science.

Mr H. H. EMSLEY, B.Sc., Vice-President, in the Chair.

Elections: (Membership) Miss H. G. CONRADY and Dr S. ENGLISH.
(Fellowship) Col. Sir GERALD P. LENOX-CONYNGHAM, Mrs L. M. LOCK, Dr H. MOORE, and Dr D. S. PERFECT.

The following paper was read:

"Microscopy of the Living Eye." By BASIL GRAVES, M.C., M.A., F.A.C.S., M.R.C.S.

Messrs Clement Clarke, Ltd., exhibited an ophthalmoscope attachment to the Fincham slit lamp.

Messrs Cambridge Instrument Co., Ltd., exhibited the Cambridge photographic ophthalmoscope (Salomonson pattern).

7th April, 1927.

Meeting held at the Imperial College of Science.

Dr R. S. CLAY, President, in the Chair.

Elections: (Membership) Mr D. A. BROOKS, Dr A. S. BURGESS, and Mr W. B. STOKES.

The following papers were read:

"Huygens' Principle and the Phenomena of Total Reflexion." By Prof. C. V. RAMAN, F.R.S. (Read by Mr E. T. Hanson.)

"The Hartmann Formula for the Dispersion of Glass." By H. W. LEE, B.A. (Read by Mr J. W. Perry.)

12th May, 1927.

Meeting held at the Imperial College of Science.

Dr R. S. CLAY, President, in the Chair.

Election: (Student Membership) Mr T. H. WINSER.

The following paper was read:

"Contributions to the History of English Opticians in the First Half of the Nineteenth Century (with Special Reference to Spectacle History)." By Prof. MORITZ VON ROHR, Ph.D., Hon. M.D.

9th June, 1927.

Meeting held at the Imperial College of Science.

Mr T. SMITH, M.A., Past President, in the Chair.

The following papers were read:

"The Use of Photo-electric Cells for the Photometry of Electric Lamps." By T. H. HARRISON, Ph.D., B.Sc., A.Inst.P.

"An Experimental Study of the Best Minimum Wave-length for Visual Achromatism." By R. KINGSLAKE, A.R.C.S., D.I.C., M.Sc., F.R.A.S.

"On Brewster's Bands." Part II. By SUSHIL KRISHNA DATTA, M.Sc., A.Inst.P. (Read in title only.)

THE OPTICAL SOCIETY

REPORT OF THE COUNCIL TO THE ANNUAL MEETING OF MEMBERS, 10TH FEBRUARY, 1927

THE COUNCIL of the Optical Society presents the following Report for the year ending 31st December, 1926:

During the year the Annual General Meeting and nine other Meetings of the Society were held at the Imperial College of Science and Technology by permission of the Governors and Professor Callendar, to whom the thanks and appreciation of the Society are due.

At the January meeting three important papers on colorimetry were communicated by Mr J. Guild. These papers, entitled "A Trichromatic Colorimeter suitable for Standardisation Work," "On a New Method of Colorimetry," and "A Criticism of the Monochromatic-plus-white Method of Colorimetry," are published in Volume 27, No. 2, of the *Transactions*. In June Professor M. von Rohr presented a paper on "Fraunhofer's Work and its Present-day Significance" which he dedicated to the memory of Fraunhofer's death at Munich on 7th June, 1826.

The meeting in February was devoted entirely to ophthalmological papers arranged by the sub-committee referred to in the Report for last year. The Council trusts that the interest of members on this side of optics will justify the formation of an Ophthalmological Section.

Early in the year the Council decided to elect the President for the time being of the Optical Society of America an *ex-officio* Fellow, and at the November meeting it was announced that the American Society had returned the compliment by creating the President for the time being of the Optical Society an Honorary Member *ex-officio*.

In conjunction with the Physical Society of London, the Sixteenth Annual Exhibition of Electrical, Optical and other Physical Apparatus was held at the Imperial College of Science and Technology on the 5th, 6th, and 7th of January and was attended by a greater number than in previous years.

New features of the Exhibition were the inclusion of a Research and Experimental Section apart from the usual Trade Exhibits and the extension of the Exhibition to three days, the last day being devoted to the public as a free service.

During the Exhibition discourses were delivered by Mr J. E. Barnard on "The Search for Ultra-Microscopic Organisms," by Professor A. F. C. Pollard on "The Mechanical Design of Instruments," and by Major W. S. Tucker on "Electrical Listening."

The Optical Convention to which reference was made in the last Report was held during the week commencing 17th April at the Imperial College of Science and Technology with marked success, and it is confidently hoped that it will have

been of assistance to the optical industry of the country. As an exceptional number of papers were read and discussed at the meeting the cost of publishing the Proceedings of the Convention largely exceeded the sum originally anticipated, with the result that a considerable portion of the Guarantee Fund will be required to meet the deficit.

The members of the Optical Society were invited by the Board of the Institute of Physics to attend the lectures on "Physics in Industry" and the thanks of the Society are due to the Board for their courtesy.

The thanks of the Society are also due to the Illuminating Engineering Society, who, as in previous years, have issued invitations to all their meetings.

With the sanction of the Society at the Annual General Meeting, the President, Mr T. Smith, and the Hon. Treasurer, Major Henrici, were appointed Trustees of the Society's funds.

Mr R. S. Whipple has continued as the representative of the Society on the National Committee for Physics, and Instructor-Commander T. Y. Baker has represented the Society on the Board of the Institute of Physics.

It has been felt for some time that a detailed subject index to the whole of the *Transactions* was needed, and after careful consideration by a Committee the Council decided to publish this index arranged on the bibliographical system of the Institut International de Bibliographie. This entailed not only the publication of the index but of a descriptive "Manual" entitled *The Decimal Bibliographical Classification of the Institut International de Bibliographie, partly translated for the formation and use of a Universal Bibliographical Repertory concerning Optics, Light, and Cognate Subjects.*

The Subject Index of the *Transactions*, Volumes I–XXV, was published early in the current year, and the Subject Index to Volumes XXVI and XXVII has been recently published. In future every Part of the *Transactions* will contain in duplicate a Subject Index to the contents, printed upon one side of a separate sheet.

A duplicate copy of these indexes may be utilised for the purpose of forming a separate Author Index to the papers published by the Society.

As far as the Council is aware this cumulative and comprehensive index is the first attempt of any learned Society to arrange and group the contents of its published papers in accordance with the methods of the Institut International de Bibliographie, and the Council is much indebted to Professor Pollard for undertaking the very laborious work involved in the preparation of these indexes and the "Manual."

LIBRARY

In addition to the regular Journals the following publications have been received during 1926:

Anniversary Volume dedicated to Professor Hantaro Nagaoka by his friends and pupils on the completion of twenty-five years of his Professorship.

Helmholtz's Treatise on Physiological Optics. Vol. 3, *The Perceptions of Vision*. Translated from the Third German Edition. Edited by Prof. JAMES P. C. SOUTHALL.

Fremdsprachliches Optisches Wörterbuch (*deutsch-englisch-französisch-spanisch*). Teil II: *Die Brille*. Bearbeitet von Dr W. Pietzsch.

Stereoscopic Photography. By ARTHUR W. JUDGE.

The Principles of Physical Optics. By ERNST MACH. Translated by JOHN S. ANDERSON, M.A., D.Sc., Ph.D., and A. F. A. YOUNG, B.Sc., Ph.D.

Commission Internationale de l'Éclairage: Recueil des Travaux et Compte Rendu des Séances, Sixième Session, Genève, Juillet, 1924.

Measurements of the Index of Refraction of Glass at High Temperatures. By C. G. PETERS.

Photometry. By JOHN W. T. WALSH, M.A., M.Sc., A.M.I.E.E., F.Inst.P.

Experimental Study of the Relation between Intermittent and Non-intermittent Sector-Wheel Photographic Exposures. By RAYMOND DAVIS.

Microscopical Illuminating Apparatus (Catalogue). By Messrs OGILVY & CO.

The Pictorial Annual of the Royal Photographic Society of Great Britain, 1926, *and a Critical Dissertation*. By F. C. TILNEY, F.R.P.S.

Department of Scientific and Industrial Research: Illumination Research, Technical Papers:

No. 1. *The Terminology of Illumination and Vision*.

No. 2. *The Transmission Factor of Commercial Window Glasses*. By A. K. TAYLOR, A.C.G.I., M.I.E.E., A.M.I.C.E., and C. J. W. GRIEVESON, M.A., B.Sc.

No. 3. *Light Distribution from Industrial Reflector Fitting No.* 1 (*British Engineering Standard*). By H. BUCKLEY, B.Sc., and C. J. W. GRIEVESON, M.A., B.Sc.

Field Astronomy for Engineers and Surveyors. By DAVID CLARK, M.A., B.Sc., M.Inst.C.E.

Wireless Pictures and Television. By T. THORNE BAKER.

Penrose's Annual: The Process Year Book and Review of the Graphic Arts, Vol. 28. Edited by WM. GAMBLE, F.R.P.S., F.O.S.

Recent Developments in Lamp Life-testing Equipment and Methods. By J. F. SKOGLAND and R. P. TEELE, Jr.

Report of the British Association for the Advancement of Science, 1925.

Annual Report of the Smithsonian Institution for 1924.

Tables Annuelles Internationales de Constantes et Données Numériques. Données Numériques de Spectroscopie, 1925.

Report of Proceedings of the Second Conference of the Association of Special Libraries and Information Bureaux.

Columbia University: Professional courses in Optometry, 1926–1927.

Additions to the Catalogue, October 1926, *Clifford's Inn Hall Library*.

MEMBERSHIP

During the year one *ex-officio* Fellow, twenty Members and three Student Members were elected. Nine members resigned, seven were removed, and two died. There is therefore an increase of six in the membership of the Society, leaving the number of members on the 1st January, 1927, as follows:

Ex-officio Fellow	1
Honorary Fellows	10
Fellows	103
Fellows non-resident in British Isles	9
Members	131
Members non-resident in British Isles	26
Corporate Members	9
Corporate Members non-resident in British Isles	2
Student Members	8
Total	299

TREASURER'S REPORT FOR THE YEAR 1926

THE income for 1926 is less than in recent years owing to the reductions in the subscriptions which came into force in that year. The decrease is, however, less than might have been anticipated as there has been a satisfactory increase in the membership. At the same time it is important for the well-being of the Society that the membership should increase still further. The income includes a sum of £21 received from the Physical Society on account of the joint exhibition held in January. The charges to exhibitors are intended to balance the expenses of the exhibition, but in previous years have failed to do so, the loss being met by the two Societies. This receipt may be set against the previous payments and the Society's share of the expenses of the Optical Convention.

The expenses of the year include the cost of the decimal index to the first 25 volumes of the *Transactions*, and the "Manual" describing the system adopted. This amounted to £264, and is a non-recurrent expenditure some of which it is hoped will be recouped by sales. The normal expenditure for the year amounts to £775 and is somewhat above the average, mainly owing to an increase in the size of the *Transactions* and the cost of printing the new rules. The normal expenditure exceeds the income by only £22. 10*s*.

The whole of the £200 mentioned in last year's report as guaranteed and lent to the Optical Convention still appears as an asset on the Balance Sheet, as the accounts of the Convention have not yet been closed. Owing to the unexpected number of papers read at the Convention, the expense of printing its *Proceedings* proved greater than was anticipated, and there will undoubtedly be a considerable call on the Guarantee Fund.

It was necessary, as had been foreseen, to realise some of the investments of the Society in order to pay for the index, and the securities sold realised £27 less than their cost price, at which they appeared in the previous Balance Sheets. This loss was mostly due to a bond purchased before the war.

The books and manuscripts of the Society are insured against loss by fire with the Guardian Assurance Co., Ltd., for the sum of £300, but they are not included in the assets of the Society shown in the Balance Sheet.

E. O. HENRICI,
Hon. Treasurer.

10th January, 1927.

INCOME AND EXPENDITURE ACCOUNT FOR THE YEAR ENDING 31ST DECEMBER, 1926

Dr EXPENDITURE

	£ s. d.	£ s. d.
To Printing and Stationery:		
Printing and distributing *Transactions*	544 18 6	
Special Index and Manual	264 6 0	
Office printing and stationery	39 9 6½	
		848 14 0½
Clerical assistance:		
Editorial expenses	58 18 4	
Other expenses	42 12 0	
		101 10 4
Postage and receipt stamps		45 3 7½
Sundry expenses		24 3 8½
Advertising		19 13 0
Life compositions account		21 0 0
		£1060 4 3½

Cr INCOME

	£ s. d.	£ s. d.
By Subscriptions:		
Fellows, arrears	12 12 0	
" for 1926	216 6 0	
Members, arrears	24 3 0	
" for 1926	156 9 0	
Corporate members	34 13 0	
Students	4 4 0	
	448 7 0	
Less rebate to members of the Institute of Physics	11 11 11	
		436 15 1
Advertisements in *Transactions*, *less* expenses		35 7 6
Sales of publications		212 3 7
Interest on investments and deposit		47 1 6
Physical Society—Exhibition account		21 0 0
Life compositions		21 0 0
Balance—Expenditure over Income		286 16 7½
		£1060 4 3½

BALANCE SHEET AT 31ST DECEMBER, 1926

LIABILITIES

	£ s. d.	£ s. d.
To Sundry Creditors		218 6 3
Accumulated Funds:		
Balance at 31st Dec. 1925	1001 0 1	
Less expenditure over income	286 16 7½	
Less loss on realising securities	27 4 11	
		686 18 6½
Life compositions account:		
Balance at 31st Dec. 1925	52 10 0	
Plus received in 1926	21 0 0	
		73 10 0
Subscriptions for 1927 received in advance		4 0 11
		£982 15 8½

ASSETS

	£ s. d.
By Sundry Debtors	55 10 4
Optical Convention	200 0 0
Cash at Bank	46 18 3
Cash in hand	9 9 9
Petty cash in hand	13 11 1½
Investments at cost*	657 6 3
	£982 15 8½

* *Mem.* Value of Investments, market price, 31 Dec. 1926 is £655. 13*s.* 9*d.*

E. O. HENRICI,
Hon. Treasurer.

We, the undersigned Auditors elected at the General Meeting held on 16th December, 1926, do hereby certify that we have examined the above accounts and do find them correct.

C. J. BECK,
R. W. CHESHIRE,
Hon. Auditors.

7th February, 1927.

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